GLOBAL REGULARITY OF CRITICAL SCHRÖDINGER MAPS: SUBTHRESHOLD DISPERSED ENERGY

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Abstract. We consider the Schrödinger map initial value problem

$$\begin{cases} \partial_t \phi & = & \phi \times \Delta \phi \\ \phi(x, 0) & = & \phi_0(x), \end{cases}$$

with $\phi_0 : \mathbf{R}^2 \to \mathbf{S}^2 \hookrightarrow \mathbf{R}^3$ a smooth H_Q^∞ map from the Euclidean space $\mathbf{R}^2$ to the sphere $\mathbf{S}^2$. Given energy-dispersed data ϕ_0 with subthreshold energy, we prove that the Schrödinger map system admits a unique global smooth solution. Also shown are global-in-time bounds on certain Sobolev norms of ϕ. This improves earlier analogous conditional results [43]. Toward establishing global regularity and global bounds, we prove refined local smoothing and bilinear Strichartz estimates for a certain class of magnetic nonlinear Schrödinger equations by modifying the physical-space method used in [43], which in turn was inspired by the Planchon-Vega approach [40]. Our improvement of the conditional results of [43] is achieved in large part through a refined analysis of the caloric gauge; in particular, we pay special attention to the commutator of the Schrödinger map and harmonic map heat flows.

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1. Introduction

We consider the Schrödinger map initial value problem

$$\begin{cases} \partial_t \phi & = & \phi \times \Delta \phi \\ \phi(x, 0) & = & \phi_0(x), \end{cases} \tag{1.1}$$

with $\phi_0 : \mathbf{R}^d \to \mathbf{S}^2 \hookrightarrow \mathbf{R}^3$.

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The system (1.1) enjoys conservation of energy

$$E(\phi(t)) := \frac{1}{2} \int_{\mathbf{R}^d} |\partial_x \phi(t)|^2 dx \tag{1.2}$$

and mass

$$M(\phi(t)) := \int_{\mathbf{R}^d} |\phi(t) - Q|^2 dx.$$

When $d = 2$, both (1.1) and (1.2) are invariant with respect to the scaling

$$\phi(x,t) \to \phi(\lambda x, \lambda^2 t), \qquad \lambda > 0, \tag{1.3}$$

and so when $d = 2$ the system (1.1) is *energy-critical*. In our study we shall restrict ourselves to the energy-critical setting.

For the physical significance of (1.1), see [9, 37, 38, 31]. The system also arises naturally from the (scalar-valued) free linear Schrödinger equation

$$(\partial_t + i\Delta)u = 0$$

by replacing the target manifold $\mathbf{C}$ with the sphere $\mathbf{S}^2 \hookrightarrow \mathbf{R}^3$, which then requires replacing Δu with $(u^*\nabla)_j \partial_j u = \Delta u - \perp(\Delta u)$ and i with the complex structure $u \times \cdot$. Here $\perp$ denotes orthogonal projection onto the normal bundle, which, for a given point (x,t), is spanned by $u(x,t)$. For more general analogues of (1.1), e.g., for Kähler targets other than $\mathbf{S}^2$, see [12, 33, 36]. See also [24, 25, 3] for connections with other spin systems.

The local theory is developed in [49, 9, 12, 33]. Before stating the formulation we shall use, we introduce some notation for Sobolev spaces.

For $\sigma \in [0, \infty)$, let $H^\sigma = H^\sigma(\mathbf{R}^2)$ denote the usual Sobolev space of complex-valued functions on $\mathbf{R}^2$. For any $Q \in \mathbf{S}^2$, set

$$H_Q^\sigma := \{f : \mathbf{R}^2 \to \mathbf{R}^3 \text{ such that } |f(x)| \equiv 1 \text{ a.e. and } f - Q \in H^\sigma\}.$$

This is a metric space with induced distance $d_Q^\sigma(f,g) = \|f - g\|_{H^\sigma}$. For $f \in H_Q^\sigma$ we set $\|f\|_{H_Q^\sigma} = d_Q^\sigma(f, Q)$ for short. We also define the spaces

$$H^\infty := \bigcap_{\sigma \in \mathbf{Z}_+} H^\sigma \quad \text{and} \quad H_Q^\infty := \bigcap_{\sigma \in \mathbf{Z}_+} H_Q^\sigma.$$

We use $\mathbf{Z}_+ = \{0, 1, 2 \ldots\}$ to denote the nonnegative integers.

These definitions may be naturally extended to any spacetime slab $\mathbf{R}^2 \times [-T, T]$, $T \in (0, \infty)$. For any $\sigma, \rho \in \mathbf{Z}_+$, let $H^{\sigma,\rho}(T)$ denote the Sobolev space of complex-valued functions on $\mathbf{R}^2 \times [-T, T]$ with the norm

$$\|f\|_{H^{\sigma,\rho}(T)} := \sup_{t \in (-T,T)} \sum_{\rho'=0}^{\rho} \|\partial_t^{\rho'} f(\cdot, t)\|_{H^\sigma},$$

and for $Q \in \mathbf{S}^2$ endow

$$H_Q^{\sigma,\rho} := \{f : \mathbf{R}^2 \times [-T, T] \to \mathbf{R}^3 \text{ such that } |f(x,t)| \equiv 1 \text{ a.e. and } f - Q \in H^{\sigma,\rho}(T)\}$$

with the metric induced by the $H^{\sigma,\rho}(T)$ norm. Also, define the spaces

$$H^{\infty,\infty}(T) = \bigcap_{\sigma,\rho\in\mathbf{Z}_+} H^{\sigma,\rho}(T) \quad \text{and} \quad H_Q^{\infty,\infty}(T) = \bigcap_{\sigma,\rho\in\mathbf{Z}_+} H_Q^{\sigma,\rho}(T).$$

For $f \in H^\infty$ and $\sigma \geq 0$ we define the homogeneous Sobolev norms as

$$\|f\|_{\dot{H}^\sigma} = \|\hat{f}(\xi) \cdot |\xi|^\sigma\|_{L^2}.$$

We use $\hat{f}$ to denote the Fourier transform in the spatial variables only, i.e.,

$$\hat{f}(\xi, t) := \int e^{-2\pi i x\cdot\xi} f(x,t)dx.$$

Theorem 1.1 (Local existence and uniqueness)**.** *If the initial data $\phi_0 \in \mathbf{R}^d$ is such that $\phi_0 \in H_Q^\infty$ for some $Q \in \mathbf{S}^2$, then there exists a time $T = T(\|\phi_0\|_{H_Q^{25}}) > 0$ for which there exists a unique solution $\phi \in C([-T,T] \to H_Q^\infty)$ of the initial value problem (1.1).*

For global results in the $d = 1$ setting, see [9, 41]. For $d \geq 3$, see [6, 7, 2, 19, 21]. Concerning the related modified Schrödinger map system, see [22, 23, 36].

In the energy-critical $d = 2$ setting there is what is known as the *threshold conjecture*, which asserts that global wellposedness holds for (1.1) given initial data with energy below a certain energy threshold, and that finite-time blowup is possible for certain initial data with energy above this threshold. The threshold is directly tied to the nontrivial stationary solutions of (1.1), i.e., maps ϕ into $\mathbf{S}^2$ that satisfy

$$\phi \times \Delta\phi \equiv 0$$

and that do not send all of $\mathbf{R}^2$ to a single point of $\mathbf{S}^2$. Therefore we identify such stationary solutions with nontrivial harmonic maps $\mathbf{R}^2 \to \mathbf{S}^2$, which we refer to as *solitons* for (1.1). It turns out that there exist no nontrivial harmonic maps into the sphere $\mathbf{S}^2$ with energy less than 4π, and that the harmonic map given by the inverse of stereographic projection has energy precisely equal to $4\pi =: E_{\text{crit}}$. We therefore refer to the range of energies $[0, E_{\text{crit}})$ as *subthreshold*, and call E_{crit} the *critical* or *threshold energy*.

Recently, an analogous threshold conjecture was established for wave maps (see [30, 42, 46, 47] and, for hyperbolic space, [29, 54, 55, 56, 57, 58]). When $\mathcal{M}$ is a hyperbolic space, or, as in [46, 47], a generic compact manifold, we may define the associated energy threshold $E_{\text{crit}} = E_{\text{crit}}(\mathcal{M})$ as follows. Given a target manifold $\mathcal{M}$, consider the collection $\mathcal{S}$ of all non-constant finite-energy harmonic maps $\phi : \mathbf{R}^2 \to \mathcal{M}$. If this set is empty, as is, for instance, the case when $\mathcal{M}$ is equal to a hyperbolic space $\mathbf{H}^m$, then we formally set $E_{\text{crit}} = +\infty$. If $\mathcal{S}$ is nonempty, then it turns out that the set $\{E(\phi) : \phi \in \mathcal{S}\}$ has a least element and that, moreover, this energy value is positive. In such case we call this least energy E_{crit}. The threshold E_{crit}

depends upon geometric and topological properties of the target manifold $\mathcal{M}$; see [32, Chapter 6] for further discussion. This definition yields $E_{\mathrm{crit}} = 4\pi$ in the case of the sphere $\mathbf{S}^2$. For further discussion of the critical energy level in the wave maps setting, see [47, 54].

We now summarize what is known for Schrödinger maps in $d = 2$. Asymptotic stability of harmonic maps of topological degree $|m| \geq 4$ under the Schrödinger flow is established in [16]. The result is extended to maps of degree $|m| \geq 3$ in [17]. A certain energy-class instability for degree-1 solitons of (1.1) is shown in [5], where it is also shown that global solutions always exist for small localized equivariant perturbations of degree-1 solitons. Finite-time blowup for (1.1) is demonstrated in [34, 35], using less-localized equivariant perturbations of degree-1 solitons, thus resolving the blowup assertion of the threshold conjecture. Global wellposedness for solutions with small critical Sobolev norm (in all dimensions $d \geq 2$) is shown in [4]. The precursor [43] of the present article establishes a conditional global result given energy dispersed data. In the radial setting (which excludes harmonic maps), [15] establishes global wellposedness at any energy level. Most recently, [1] establishes global existence and uniqueness as well as scattering given 1-equivariant data with energy less than 4π. Although stating the results only for data with energy less than 4π, [1] also notes that their proofs remain valid for maps with energy slightly larger than 4π, suggesting that the "right" threshold conjecture for Schrödinger maps should be stated in terms of homotopy class, leading to a threshold of 8π rather than 4π in the case where the target is $\mathbf{S}^2$. See the Introduction of [1] for further discussion of this point.

The main purpose of this article is to show that (1.1) admits a unique smooth global solution ϕ given smooth subthreshold initial data ϕ_0 satisfying a certain energy dispersion hypothesis. In terms of Besov spaces, the energy dispersion hypothesis on ϕ_0 may be expressed as $\|\phi_0\|_{\dot{B}^1_{2,\infty}} \leq \varepsilon$ for ε sufficiently small. In this sense our result may be interpreted as a Schrödinger-map analogue of a global result for the cubic NLS found in [8], where it is shown that the cubic NLS on $\mathbf{R}^2$ admits a unique global solution given initial data u_0 sufficiently small in $\dot{B}^0_{2,\infty}$. Instead of introducing Besov spaces, we opt to formulate the energy dispersion hypothesis directly in terms of norms of Littlewood-Paley localizations. Our notation for a standard Littlewood-Paley frequency localization of a function f to frequencies $\sim 2^k$ is $P_k f$, and to frequencies $\lesssim 2^k$, $P_{\leq k} f$. A precise definition is given in §2.

We now state our main global result.

Theorem 1.2 (Global regularity)**.** *Fix $Q \in \mathbf{S}^2$. Then there exists $\varepsilon_0 > 0$ such that for all $\phi_0 \in H_Q$ with $E_0 := E(\phi_0) < E_{\mathrm{crit}}$ and*

$$\sup_{k \in \mathbf{Z}} \|P_k \partial_x \phi_0\|_{L^2_x} \leq \varepsilon_0,$$

equation (1.1) admits a unique global solution $\phi \in C(\mathbf{R} \to H_Q^\infty)$.

We remark that it may be possible to drop the subthreshold assumption on the initial data. The obstruction appears only to be in defining the gauge that we wish to use. In particular, we adopt the caloric gauge, which was introduced in [52] and then developed in the subthreshold setting for a general class of target manifolds in [44]. In adapting that construction to a dispersed energy setting, the crucial modification would have to be to the minimal blowup solution argument of [44, §6]. Restricting to maps with sufficient energy dispersion in order to rule out harmonic map formation, for instance, might constitute an effective replacement for the subthreshold energy assumption in that work. We do not pursue this here.

Before stating the uniform bounds, we introduce some standard asymptotic notation. We use $f \lesssim g$ to denote the estimate $|f| \leq C|g|$ for an absolute constant $C > 0$. As usual, the constant is allowed to change line-to-line. To indicate the dependence of the implicit constant upon parameters (which, for instance, can include functions), we use subscripts, e.g. $f \lesssim_k g$. As an equivalent alternative we write $f = O(g)$ (or, with subscripts, $f = O_k(g)$, for instance) to denote $|f| \leq C|g|$. If both $f \lesssim g$ and $g \lesssim f$ hold, then we indicate this by writing $f \sim g$.

Theorem 1.3 (Uniform bounds)**.** *Fix* $Q \in \mathbf{S}^2$. *Let* $\sigma_1 \geq 1$. *Then there exists* $\tilde{\varepsilon}_0(\sigma_1) \in (0, \varepsilon_0]$ *such that for all* $\phi_0 \in H_Q$ *with* $E_0 := E(\phi_0) < E_{\text{crit}}$ *and*

$$\sup_{k \in \mathbf{Z}} \|P_k \partial_x \phi_0\|_{L_x^2} \leq \tilde{\varepsilon}_0,$$

the global solution ϕ *constructed in Theorem 1.2 satisfies the uniform bounds*

$$\sup_{t \in \mathbf{R}} \|\phi(t) - Q\|_{H^\sigma} \lesssim_\sigma \|\phi_0 - Q\|_{H^\sigma}, \quad 1 \leq \sigma \leq \sigma_1.$$

In [43] we proved conditional versions of Theorems 1.2 and 1.3. The condition required there is a bound on a certain $L_{t,x}^4$ norm of the derivative of the solution ϕ. In view of what is known about simpler nonlinear Schrödinger equations, this condition seems rather strong. Nevertheless, even when equipped with this bound, proving global regularity is far from trivial.

In order to go beyond the small-energy results of [4], [43] introduces physical-space proofs of local smoothing and bilinear Strichartz estimates in the spirit of [39, 40, 59]. These proofs have the virtue of being less dependent upon perturbative methods. The bilinear Strichartz estimate in [43] is a nonlinear analogue of the improved bilinear Strichartz estimate of [8]. Proofs of both the local smoothing estimate and the bilinear Strichartz estimate naturally account for magnetic nonlinearities, and the technique developed should be applicable to other settings. Smoothing and Strichartz estimates have been and continue to be very active areas of research. For local smoothing in

the context of Schrödinger equations, see [26, 27, 28, 18, 19, 21] and the references therein. For other Strichartz and smoothing results for magnetic Schrödinger equations, see [45, 11, 13, 14, 10].

In the present article we streamline the proofs of the local smoothing and bilinear Strichartz estimates introduced in [43] with the intention of making the main ideas more transparent. The key improvement that allows us to remove the "conditional" aspect of the results in [43], however, rests upon showing that the "nonmagnetic" nonlinearity is perturbative. This improved bound is partially achieved through a slight technical modification of the main functions spaces introduced in [4] and used in [43]. The more significant contribution, though, comes from a refined analysis of the behavior of the caloric gauge, which is taken up in §4.2.

Below we give a high-level overview of the proofs of Theorems 1.2 and 1.3 as well as a guide to how the remainder of the article is organized. However we first introduce a few more tools and definitions.

The following standard *Bernstein inequalities*, with $\sigma \geq 0$ and $1 \leq p \leq q \leq \infty$, will be used frequently:

$$\begin{aligned}
\|P_{\leq k}|\partial_x|^\sigma f\|_{L^p_x(\mathbf{R}^2)} &\lesssim_{p,\sigma} 2^{\sigma k}\|P_{\leq k} f\|_{L^p_x(\mathbf{R}^2)} \\
\|P_k|\partial_x|^{\pm\sigma} f\|_{L^p_x(\mathbf{R}^2)} &\lesssim_{p,\sigma} 2^{\pm\sigma k}\|P_k f\|_{L^p_x(\mathbf{R}^2)} \\
\|P_{\leq k} f\|_{L^q_x(\mathbf{R}^2)} &\lesssim_{p,q} 2^{2k(1/p-1/q)}\|P_{\leq k} f\|_{L^p_x(\mathbf{R}^2)} \\
\|P_k f\|_{L^q_x(\mathbf{R}^2)} &\lesssim_{p,q} 2^{2k(1/p-1/q)}\|P_k f\|_{L^p_x(\mathbf{R}^2)}.
\end{aligned}$$

Frequency envelopes provide us with a way to rigorously manage "frequency leakage" phenomenon and frequency cascades produced by nonlinear interactions. A parameter δ appears in the definition; for the purposes of this paper $\delta = \frac{1}{64}$ suffices.

Definition 1.4 (Frequency envelopes)**.** A positive sequence $\{a_k\}_{k\in\mathbf{Z}}$ is a *frequency envelope* if it belongs to ℓ^2 and is slowly varying:

$$a_k \leq a_j 2^{\delta|k-j|}, \qquad j,k \in \mathbf{Z}. \tag{1.4}$$

A frequency envelope $\{a_k\}_{k\in\mathbf{Z}}$ is ε-energy dispersed if it satisfies the additional condition

$$\sup_{k\in\mathbf{Z}} a_k \leq \varepsilon.$$

Note in particular that frequency envelopes satisfy the following summation rules:

$$\sum_{k'\leq k} 2^{pk'} a_{k'} \lesssim (p-\delta)^{-1} 2^{pk} a_k \qquad p > \delta \tag{1.5}$$

$$\sum_{k'\geq k} 2^{-pk'} a_{k'} \lesssim (p-\delta)^{-1} 2^{-pk} a_k \qquad p > \delta. \tag{1.6}$$

In practice we work with p bounded away from δ, e.g., $p > 2\delta$ suffices, and iterate these inequalities only $O(1)$ times. Therefore in applications we drop the factors $(p-\delta)^{-1}$ appearing in (1.5) and (1.6).

Given initial data $\phi_0 \in H_Q^\infty$, define for all $\sigma \geq 0$ and $k \in \mathbf{Z}$

$$c_k(\sigma) := \sup_{k' \in \mathbf{Z}} 2^{-\delta|k-k'|} 2^{\sigma k'} \|P_{k'} \partial_x \phi_0\|_{L_x^2}. \tag{1.7}$$

Set $c_k := c_k(0)$ for short. For $\sigma \in [0, \sigma_1]$ it then holds that

$$\|\partial_x \phi_0\|_{\dot{H}_x^\sigma}^2 \sim \sum_{k \in \mathbf{Z}} c_k^2(\sigma) \qquad \text{and} \qquad \|P_k \partial_x \phi_0\|_{L_x^2} \leq c_k(\sigma) 2^{-\sigma k}. \tag{1.8}$$

Similarly, for $\phi \in H_Q^{\infty,\infty}(T)$, define for all $\sigma \geq 0$ and $k \in \mathbf{Z}$

$$\gamma_k(\sigma) := \sup_{k' \in \mathbf{Z}} 2^{-\delta|k-k'|} 2^{\sigma k'} \|P_{k'} \phi\|_{L_t^\infty L_x^2}. \tag{1.9}$$

Set $\gamma_k := \gamma_k(1)$.

For technical reasons related to discretization (see §3), it will be convenient to construct a solution ϕ on a time interval $(-2^{2\mathcal{K}}, 2^{2\mathcal{K}})$ for some given $\mathcal{K} \in \mathbf{Z}_+$ and then proceed to prove bounds that are uniform in $\mathcal{K}$. We assume $1 \ll \mathcal{K} \in \mathbf{Z}_+$ is chosen and hereafter fixed. Invoking Theorem 1.1, we assume that we have a solution $\phi \in C([-T, T] \to H_Q^\infty)$ of (1.1) on the time interval $[-T, T]$ for some $T \in (0, 2^{2\mathcal{K}}]$. In order to extend ϕ to a solution on all of $(-2^{2\mathcal{K}}, 2^{2\mathcal{K}})$ with uniform bounds (uniform in $T, \mathcal{K}$) it suffices to prove uniform a priori estimates on

$$\sup_{t \in [-T,T]} \|\phi(t)\|_{H_Q^\sigma}$$

for, say, σ in the interval $[1, \sigma_1]$, with $\sigma_1 \gg 1$ chosen sufficiently large (e.g., $\sigma_1 = 25$ will do).

The first step in our approach, carried out in §2, is to lift the Schrödinger map system (1.1) to the tangent bundle and view it with respect to the caloric gauge. The lift of (1.1) becomes a magnetic nonlinear Schrödinger equation of the form

$$(i\partial_t + \Delta)\psi_m = B_m + V_m, \tag{1.10}$$

with initial data $\psi_m(0)$. Here B_m and V_m respectively denote the magnetic and electric potentials (see (2.8) and (2.9) for definitions).

The goal then becomes proving a priori bounds on $\|\psi_m\|_{L_t^\infty H_x^\sigma}$. To establish these bounds, we in fact prove stronger frequency-localized estimates. Finally, to complete the proof of Theorem 1.2, we must transfer the a priori bounds on the derivative fields ψ_m back to bounds on the map ϕ.

Energy spaces are not sufficient for controlling $P_k \psi_m$, and so we combine in addition to these a host of Strichartz, local smoothing, and maximal function type spaces into one space $G_k(T)$ (see §3) for precise definitions). As our

goal will be to express control of the $G_k(T)$ norms of $P_k\psi_m$ in terms of the energy of the frequency localizations of the initial data, we introduce the following frequency envelopes. Let $\sigma_1 \in \mathbf{Z}_+$ be positive. For $\sigma \in [0, \sigma_1 - 1]$, set

$$b_k(\sigma) = \sup_{k' \in \mathbf{Z}} 2^{\sigma k'} 2^{-\delta|k-k'|} \| P_{k'} \psi_x \|_{G_k(T)}. \tag{1.11}$$

By (2.32) these envelopes are finite and in ℓ^2. We abbreviate $b_k(0)$ by setting $b_k := b_k(0)$.

We now state the key result for solutions of the gauge field equation (1.10).

Theorem 1.5. *Assume $T \in (0, 2^{2\mathcal{K}}]$ and $Q \in S^2$. Choose $\sigma_1 \in \mathbf{Z}_+$ positive. Let $\varepsilon_0 > 0$ and let $\phi \in H_Q^{\infty,\infty}(T)$ be a solution of the Schrödinger map system (1.1) whose initial data ϕ_0 has energy $E_0 := E(\phi_0) < E_{\mathrm{crit}}$ and satisfies the energy dispersion condition*

$$\sup_{k \in \mathbf{Z}} c_k \leq \varepsilon_0. \tag{1.12}$$

Suppose that the bootstrap hypothesis

$$b_k \leq \varepsilon_0^{-1/10} c_k \tag{1.13}$$

is satisfied. Then, for ε_0 sufficiently small,

$$b_k(\sigma) \lesssim c_k(\sigma) \tag{1.14}$$

holds for all $\sigma \in [0, \sigma_1 - 1]$ and $k \in \mathbf{Z}$.

We use a continuity argument to prove Theorem 1.5. For $T' \in (0, T]$, let

$$\Psi(T') = \sup_{k \in \mathbf{Z}} c_k^{-1} \| P_k \psi_m(s = 0) \|_{G_k(T')}.$$

Then $\Psi : (0, T] \to [0, \infty)$ is well-defined, increasing, continuous, and satisfies

$$\lim_{T' \to 0} \Psi(T') \lesssim 1.$$

The critical implication to establish is

$$\Psi(T') \leq \varepsilon_1^{-1/10} \implies \Psi(T') \lesssim 1,$$

which in particular follows from

$$b_k \lesssim c_k. \tag{1.15}$$

We must also similarly establish

$$b_k(\sigma) \lesssim c_k(\sigma) \tag{1.16}$$

for $\sigma \in (0, \sigma_1 - 1]$ in order to bring under control the higher-order Sobolev norms. The bounds (1.16), however, will be seen to follow as a relatively easy consequence of the proof of inequalities (1.15), and therefore throughout this article the emphasis will be upon proving (1.15).

Corollary 1.6. *Given the conditions of Theorem 1.5,*

$$\|P_k |\partial_x|^\sigma \partial_m \phi\|_{L_t^\infty L_x^2((-T,T)\times \mathbf{R}^2)} \lesssim c_k(\sigma) \tag{1.17}$$

holds for all $\sigma \in [0, \sigma_1 - 1]$.

In view of Theorem 1.1 and (1.8), this corollary implies Theorems 1.2 and 1.3.

We provide a brief outline of the rest of the article.

In §2 we introduce the gauge field equations and the caloric gauge. We also state some basic energy estimates from [4, 43]. Proofs may be found in [43, §6].

In §3 we introduce the key function spaces. These are slight modifications of those introduced in [4]. Here we also establish the basic estimates these spaces satisfy. Finally, motivated by the off-diagonal decay in these estimates, we prove some summations that will play a key role in §4.

In §4 we paralinearize the nonlinearity and show that most of these terms are perturbative. The exception is a term schematically of the form $A_{\mathrm{lo}} \partial_x \phi_{\mathrm{hi}}$. The perturbative estimates are proven using norms that do not rely upon the main local smoothing/maximal function estimates. We take as given certain estimates from [4, 43], whose full proofs may be found in [43, §7].

In §5 we show how to manage $A_{\mathrm{lo}} \partial_x \phi_{\mathrm{hi}}$. We begin by presenting some abstract local smoothing and bilinear Strichartz estimates for magnetic nonlinear Schrödinger equations of a special form. We then show that these abstract results apply to our setting, in part thanks to the spaces we chose to employ in §4.

In §6, we bring together the results of §4 and §5 to prove (1.15). Finally, having bootstrapped bounds on the derivative fields, we show how to control the map itself.

2. Gauge field equations

In §2.1 we pass to the derivative formulation of the Schrödinger map system (1.1). All of the main arguments of our subsequent analysis take place at this level. The derivative formulation is at once both overdetermined, reflecting geometric constraints, and underdetermined, exhibiting *gauge invariance*. §2.2 introduces the caloric gauge, which is the gauge we select and work with throughout. Both [50] and [4] give good explanations justifying the use of the caloric gauge in our setting as opposed to alternatives. The reader is referred to [44] for the requisite construction of the caloric gauge for maps with energy up to E_{crit}.

2.1. **Derivative equations.** We begin with some constructions valid for any smooth function $\phi : \mathbf{R}^2 \times (-T, T) \to \mathbf{S}^2$. For a more general and extensive introduction to the gauge formalism we now introduce, see [52]. Space and time derivatives of ϕ are denoted by $\partial_\alpha \phi(x,t)$, where $\alpha = 1, 2, 3$ ranges over the spatial variables x_1, x_2 and time t with $\partial_3 = \partial_t$.

Select a (smooth) orthonormal frame $(v(x,t), w(x,t))$ for $T_{\phi(x,t)}\mathbf{S}^2$, i.e., smooth functions $v, w : \mathbf{R}^2 \times (-T,T) \to T_{\phi(x,t)}\mathbf{S}^2$ such that at each point (x,t) in the domain the vectors $v(x,t), w(x,t)$ form an orthonormal basis for $T_{\phi(x,t)}\mathbf{S}^2$. As a matter of convention we assume that v and w are chosen so that $v \times w = \phi$.

With respect to this chosen frame we then introduce the derivative fields ψ_α, setting

$$\psi_\alpha := v \cdot \partial_\alpha \phi + i w \cdot \partial_\alpha \phi. \tag{2.1}$$

Hence $\partial_\alpha \phi$ admits the representation

$$\partial_\alpha \phi = v \operatorname{Re}(\psi_\alpha) + w \operatorname{Im}(\psi_\alpha) \tag{2.2}$$

with respect to the frame (v, w). The derivative fields can be thought of as arising from the following process: First, rewrite the vector $\partial_\alpha \phi$ with respect to the orthonormal basis (v, w); then, identify $\mathbf{R}^2$ with the complex numbers $\mathbf{C}$ according to $v \leftrightarrow 1$, $w \leftrightarrow i$. This identification respects the complex structure of the target manifold.

Through this identification the Riemannian connection on $\mathbf{S}^2$ pulls back to a covariant derivative on $\mathbf{C}$, which we denote by

$$D_\alpha := \partial_\alpha + i A_\alpha.$$

The real-valued connection coefficients A_α are defined via

$$A_\alpha := w \cdot \partial_\alpha v, \tag{2.3}$$

so that in particular

$$\begin{aligned} \partial_\alpha v &= -\phi \operatorname{Re}(\psi_\alpha) + w A_\alpha \\ \partial_\alpha w &= -\phi \operatorname{Im}(\psi_\alpha) - v A_\alpha. \end{aligned}$$

Due to the fact that the Riemannian connection on $\mathbf{S}^2$ is torsion-free, the derivative fields satisfy the relations

$$D_\beta \psi_\alpha = D_\alpha \psi_\beta. \tag{2.4}$$

or equivalently,

$$\partial_\beta A_\alpha - \partial_\alpha A_\beta = \operatorname{Im}(\psi_\beta \overline{\psi_\alpha}) =: q_{\beta\alpha},$$

A straightforward calculation (which uses the fact that the sphere has constant curvature) shows

$$\partial_\beta A_\alpha - \partial_\alpha A_\beta = \operatorname{Im}(\psi_\beta \overline{\psi_\alpha}) =: q_{\beta\alpha},$$

so that the curvature of the connection is given by

$$[D_\beta, D_\alpha] := D_\beta D_\alpha - D_\alpha D_\beta = iq_{\beta\alpha}. \tag{2.5}$$

Assuming now that we are given a smooth solution ϕ of the Schrödinger map system (1.1), we derive the equations satisfied by the derivative fields ψ_α. The system (1.1) directly translates to

$$\psi_t = iD_\ell \psi_\ell \tag{2.6}$$

because

$$\phi \times \Delta\phi = J(\phi)(\phi^*\nabla)_j \partial_j \phi,$$

where $J(\phi)$ denotes the complex structure $\phi\times$ and $(\phi^*\nabla)_j$ the pullback of the Levi-Civita connection ∇ on the sphere.

Let us pause to note the following conventions regarding indices. Roman typeface letters are used to index spatial variables. Greek typeface letters are used to index the spatial variables along with time. Repeated lettered indices within the same subscript or occurring in juxtaposed terms indicate an implicit summation over the appropriate set of indices.

Using (2.4) and (2.5) in (2.6) yields

$$D_t \psi_m = iD_\ell D_\ell \psi_m + q_{\ell m} \psi_\ell,$$

which is equivalent to the nonlinear Schrödinger equation

$$(i\partial_t + \Delta)\psi_m = \mathcal{N}_m, \tag{2.7}$$

where the nonlinearity $\mathcal{N}_m$ is defined by the formula

$$\mathcal{N}_m := -iA_\ell \partial_\ell \psi_m - i\partial_\ell(A_\ell \psi_m) + (A_t + A_x^2)\psi_m - i\psi_\ell \operatorname{Im}(\overline{\psi_\ell}\psi_m).$$

We split this nonlinearity as a sum $\mathcal{N}_m = B_m + V_m$ with B_m and V_m defined by

$$B_m := -i\partial_\ell(A_\ell \psi_m) - iA_\ell \partial_\ell \psi_m \tag{2.8}$$

and

$$V_m := (A_t + A_x^2)\psi_m - i\psi_\ell \operatorname{Im}(\overline{\psi_\ell}\psi_m), \tag{2.9}$$

thus separating the essentially semilinear magnetic potential terms and the essentially quasilinear electric potential terms from each other.

We now state the gauge formulation of the differentiated Schrödinger map system:

$$\begin{cases} D_t \psi_m & = iD_\ell D_\ell \psi_m + \operatorname{Im}(\psi_\ell \overline{\psi_m})\psi_\ell \\ D_\alpha \psi_\beta & = D_\beta \psi_\alpha \\ \operatorname{Im}(\psi_\alpha \overline{\psi_\beta}) & = \partial_\alpha A_\beta - \partial_\beta A_\alpha. \end{cases} \tag{2.10}$$

A solution ψ_m to (2.10) cannot be determined uniquely without first choosing an orthonormal frame (v, w). Changing a given choice of orthonormal frame induces a gauge transformation and may be represented as

$$\psi_m \to e^{i\theta}\psi_m, \qquad A_m \to A_m + \partial_m \theta$$

in terms of the gauge components. The system (2.10) is invariant with respect to such gauge transformations.

The advantage of working with this gauge formalism rather than the Schrödinger map system or the derivative equations directly is that a carefully selected choice of gauge tames the nonlinearity. In particular, when the caloric gauge is employed, the nonlinearity in (2.7) is nearly perturbative.

2.2. **Introduction to the caloric gauge.** In this section we introduce the caloric gauge, which is the gauge we shall employ throughout the remainder of the paper. Gauges were first used to study (1.1) in [9]. We note here that the while the Coulomb gauge would seem an attractive choice, it turns out that this gauge is not well-suited to the study of Schrödinger maps in low dimension, as in low dimension parallel interactions of waves are more probable than in high dimension, resulting in unfavorable high $\times$ high $\to$ low cascades. See [50] and [4] for further discussion and a comparison of the Coulomb and caloric gauges. Also see [53, Chapter 6] for a discussion of various gauges that have been used in the study of wave maps.

The caloric gauge was introduced by Tao in [52] in the setting of wave maps into hyperbolic space. In the series of unpublished papers [54, 55, 56, 57, 58], Tao used this gauge in establishing global regularity of wave maps into hyperbolic space. In his unpublished note [50], Tao also suggested the caloric gauge as a suitable gauge for the study of Schrödinger maps. The caloric gauge was first used in the Schrödinger maps problem by Bejenaru, Ionescu, Kenig, and Tataru in [4] to establish global well-posedness in the setting of initial data with sufficiently small critical norm. We recommend [52, 55, 50, 4] for background on the caloric gauge and for helpful heuristics.

Theorem 2.1 (The caloric gauge)**.** *Let $T \in (0, \infty)$, $Q \in S^2$, and let $\phi(x,t) \in H_Q^{\infty,\infty}(T)$ be such that $\sup_{t \in (-T,T)} E(\phi(t)) < E_{\mathrm{crit}}$. Then there exists a unique smooth extension $\phi(s,x,t) \in C([0,\infty) \to H_Q^{\infty,\infty}(T))$ solving the covariant heat equation*

$$\partial_s \phi = \Delta \phi + \phi \cdot |\partial_x \phi|^2 \tag{2.11}$$

and with $\phi(0,x,t) = \phi(x,t)$. Moreover, for any given choice of a (constant) orthonormal basis (v_∞, w_∞) of $T_Q \mathbf{S}^2$, there exist smooth functions $v, w : [0,\infty) \times \mathbf{R}^2 \times (-T,T) \to S^2$ such that at each point (s,x,t), the set $\{v, w, \phi\}$ naturally forms an orthonormal basis for $\mathbf{R}^3$, the gauge condition

$$w \cdot \partial_s v \equiv 0, \tag{2.12}$$

is satisfied, and

$$|\partial_x^\rho f(s)| \lesssim_\rho \langle s \rangle^{-(|\rho|+1)/2} \tag{2.13}$$

for each $f \in \{\phi - Q, v - v_\infty, w - w_\infty\}$, multiindex ρ, and $s \geq 0$.

Proof. This is a special case of the more general main result of [44]. Whereas in [44] everything is stated in terms of the category of Schwartz functions, in fact this requirement may be relaxed to $H_Q^{\infty,\infty}(T)$ without difficulty (at least in the case of compact target manifolds) since weighted decay in L^2-based Sobolev spaces is not used in any proofs. □

We now record a couple of the technical bounds from [44], which will be useful later on in controlling terms along the heat flow.

Theorem 2.2. *The following bounds hold:*

$$\int_0^\infty \sup_{x \in \mathbf{R}^2} |\psi_x(s,x)|^2 ds \lesssim_{E_0} 1, \tag{2.14}$$

$$\|A_x(s)\|_{L_x^2(\mathbf{R}^2)} \lesssim_{E_0} 1. \tag{2.15}$$

Proof. For (2.14), see [44, §4]. For (2.15), see [44, §§7, 7.1]. □

Corollary 2.3 (Energy bounds for the frame)**.** *It holds that*

$$\|\partial_x v\|_{L_t^\infty L_x^2} \lesssim_{E_0} 1. \tag{2.16}$$

Proof. Because $|v| \equiv 1$, it holds that $v \cdot \partial_m v \equiv 0$. Therefore, with respect to the orthonormal frame (v, w, ϕ), the vector $\partial_m v$ admits the representation

$$\partial_m v = A_m \cdot w - \mathrm{Re}(\psi_m) \cdot \phi. \tag{2.17}$$

The bound (2.16) then follows from using $|w| \equiv 1 \equiv |\phi|$, $\|\psi_m\|_{L_x^2} \equiv \|\partial_m \phi\|_{L_x^2}$, energy conservation, and (2.15) all in (2.17). □

Adopting the convention $\partial_0 = \partial_s$, and now and hereafter allowing all Greek indices to range over heat time, spatial variables, and time, we define for all $(s, x, t) \in [0, \infty) \times \mathbf{R}^2 \times (-T, T)$ the various gauge components

$$\begin{aligned}
\psi_\alpha &:= v \cdot \partial_\alpha \phi + i w \cdot \partial_\alpha \phi \\
A_\alpha &:= w \cdot \partial_\alpha v \\
D_\alpha &:= \partial_\alpha + A_\alpha \\
q_{\alpha\beta} &:= \partial_\alpha A_\beta - \partial_\beta A_\alpha.
\end{aligned}$$

For $\alpha = 0, 1, 2, 3$ it holds that

$$\partial_\alpha \phi = v \, \mathrm{Re}(\psi_\alpha) + w \, \mathrm{Im}(\psi_\alpha).$$

The parallel transport condition $w \cdot \partial_s v \equiv 0$ is equivalently expressed in terms of the connection coefficients as

$$A_s \equiv 0. \tag{2.18}$$

Expressed in terms of the gauge, the heat flow (2.11) lifts to

$$\psi_s = D_\ell \psi_\ell. \tag{2.19}$$

Using (2.4) and (2.5), we may rewrite the D_m covariant derivative of (2.19) as

$$\partial_s \psi_m = D_\ell D_\ell \psi_m + i\mathrm{Im}(\psi_m \overline{\psi_\ell})\psi_\ell,$$

or equivalently

$$(\partial_s - \Delta)\psi_m = iA_\ell \partial_\ell \psi_m + i\partial_\ell(A_\ell \psi_m) - A_x^2 \psi_m + i\psi_\ell \mathrm{Im}(\overline{\psi_\ell}\psi_m). \tag{2.20}$$

More generally, taking the D_α covariant derivative, we obtain

$$(\partial_s - \Delta)\psi_\alpha = U_\alpha, \tag{2.21}$$

where we set

$$U_\alpha := iA_\ell \partial_\ell \psi_\alpha + i\partial_\ell(A_\ell \psi_\alpha) - A_x^2 \psi_\alpha + i\psi_\ell \mathrm{Im}(\overline{\psi_\ell}\psi_\alpha) \tag{2.22}$$

From (2.5) and (2.18) it follows that

$$\partial_s A_\alpha = \mathrm{Im}(\psi_s \overline{\psi_\alpha}).$$

Integrating back from $s = \infty$ (justified using (2.13)) yields

$$A_\alpha(s) = -\int_s^\infty \mathrm{Im}(\overline{\psi_\alpha}\psi_s)(s')\, ds'. \tag{2.23}$$

At $s = 0$, ϕ satisfies both (1.1) and (2.11), or equivalently, $\psi_t(s = 0) = i\psi_s(s = 0)$. While for $s > 0$ it continues to be the case that $\psi_s = D_\ell \psi_\ell$ by construction, we no longer necessarily have $\psi_t(s) = iD_\ell(s)\psi_\ell(s)$, i.e., $\phi(s, x, t)$ is not necessarily a Schrödinger map at fixed $s > 0$. In the following lemma we derive an evolution equation for the commutator $\Psi = \psi_t - i\psi_s$.

Lemma 2.4 (Flows do not commute)**.** *Set $\Psi := \psi_t - i\psi_s$. Then*

$$\begin{aligned}\partial_s \Psi &= D_\ell D_\ell \Psi + i\mathrm{Im}(\psi_t \bar{\psi}_\ell)\psi_\ell - \mathrm{Im}(\psi_s \bar{\psi}_\ell)\psi_\ell &(2.24)\\ &= D_\ell D_\ell \Psi + i\mathrm{Im}(\Psi \bar{\psi}_\ell)\psi_\ell + i\mathrm{Im}(i\psi_s \bar{\psi}_\ell)\psi_\ell - \mathrm{Im}(\psi_s \bar{\psi}_\ell)\psi_\ell. &(2.25)\end{aligned}$$

Proof. We prove (2.24), since (2.25) is a trivial consequence of it.

Applying (2.20) and (2.21) to ψ_s and ψ_t and collapsing the covariant derivative terms yields

$$\partial_s \psi_t = D_\ell D_\ell \psi_t + i\mathrm{Im}(\psi_t \overline{\psi_\ell})\psi_\ell \tag{2.26}$$

$$\partial_s \psi_s = D_\ell D_\ell \psi_s + i\mathrm{Im}(\psi_s \overline{\psi_\ell})\psi_\ell. \tag{2.27}$$

Multiply (2.27) by i to obtain the s-evolution of $i\psi_s$. Multiplication by i commutes with D_ℓ, but fails to do so with $\mathrm{Im}(\cdot)$, and thus we obtain

$$\partial_s i\psi_m = D_\ell D_\ell i\psi_s - \mathrm{Im}(\psi_s \overline{\psi_\ell})\psi_\ell. \tag{2.28}$$

Together (2.26) and (2.28) imply (2.24). □

We now record some frequency-localized energy estimates which will prove useful in controlling the paralinearized nonlinearity.

Theorem 2.5 (Frequency-localized energy estimates for heat flow)**.** *Let $f \in \{\phi, v, w\}$. Then for $\sigma \in [1, \sigma_1]$ the bound*

$$\|P_k f(s)\|_{L_t^\infty L_x^2} \lesssim 2^{-\sigma k}\gamma_k(\sigma)(1 + s2^{2k})^{-20} \tag{2.29}$$

holds, and, for any $\sigma, \rho \in \mathbf{Z}_+$, it holds that

$$\sup_{k\in\mathbf{Z}} \sup_{s\in[0,\infty)} (1+s)^{\sigma/2} 2^{\sigma k} \|P_k \partial_t^\rho f(s)\|_{L_t^\infty L_x^2} < \infty. \tag{2.30}$$

Corollary 2.6 (Frequency-localized energy estimates for the caloric gauge)**.** *For $\sigma \in [0, \sigma_1 - 1]$ it holds that*

$$\|P_k \psi_x(s)\|_{L_t^\infty L_x^2} + \|P_k A_m(s)\|_{L_t^\infty L_x^2} \lesssim 2^k 2^{-\sigma k}\gamma_k(\sigma)(1 + s2^{2k})^{-20}. \tag{2.31}$$

Moreover, for any $\sigma \in \mathbf{Z}_+$,

$$\sup_{k\in\mathbf{Z}} \sup_{s\in[0,\infty)} (1+s)^{\sigma/2} 2^{\sigma k} 2^{-k} \left(\|P_k(\partial_t^\rho \psi_x(s))\|_{L_t^\infty L_x^2} + \|P_k(\partial_t^\rho A_x(s))\|_{L_t^\infty L_x^2} \right) < \infty \tag{2.32}$$

and

$$\sup_{k\in\mathbf{Z}} \sup_{s\in[0,\infty)} (1+s)^{\sigma/2} 2^{\sigma k} \left(\|P_k(\partial_t^\rho \psi_t(s))\|_{L_t^\infty L_x^2} + \|P_k(\partial_t^\rho A_t(s))\|_{L_t^\infty L_x^2} \right) < \infty. \tag{2.33}$$

Full proofs for Theorem 2.5 and its corollary may be found in [43]. These are extensions to the energy-dispersed setting of analogous small-energy bounds from [4].

Note that Corollary 2.6 has as an elementary consequence the following:

Corollary 2.7. *For $\sigma \in [0, \sigma_1 - 1]$, It holds that*

$$\|P_k \psi_x(0, \cdot, 0)\|_{L_x^2} \lesssim 2^{-\sigma k} c_k(\sigma). \tag{2.34}$$

3. Function spaces and basic estimates

3.1. Definitions.

Definition 3.1 (Littlewood-Paley multipliers)**.** Let $\eta_0 : \mathbf{R} \to [0, 1]$ be a smooth even function vanishing outside the interval $[-8/5, 8/5]$ and equal to 1 on $[-5/4, 5/4]$. For $j \in \mathbf{Z}$, set

$$\chi_j(\cdot) = \eta_0(\cdot/2^j) - \eta_0(\cdot/2^{j-1}), \quad \chi_{\le j}(\cdot) = \eta_0(\cdot/2^j).$$

Let P_k denote the operator on $L^\infty(\mathbf{R}^2)$ defined by the Fourier multiplier $\xi \to \chi_k(|\xi|)$. For any interval $I \subset \mathbf{R}$, let χ_I be the Fourier multiplier defined by $\chi_I = \sum_{j\in I\cap\mathbf{Z}} \chi_j$ and let P_I denote its corresponding operator on $L^\infty(\mathbf{R}^2)$. We shall denote $P_{(-\infty,k]}$ by $P_{\le k}$ for short. For $\theta \in \mathbf{S}^1$ and $k \in \mathbf{Z}$, we define the operators $P_{k,\theta}$ by the Fourier multipliers $\xi \to \chi_k(\xi \cdot \theta)$.

Some frequency interactions in the nonlinearity of (2.7) can be controlled using the following Strichartz estimate:

Lemma 3.2 (Strichartz estimate)**.** *Let* $f \in L^2_x(\mathbf{R}^2)$ *and* $k \in \mathbf{Z}$*. Then the Strichartz estimate*

$$\|e^{it\Delta} f\|_{L^4_{t,x}} \lesssim \|f\|_{L^2_x}$$

holds, as does the maximal function bound

$$\|e^{it\Delta} P_k f\|_{L^4_x L^\infty_t} \lesssim 2^{k/2} \|f\|_{L^2_x}.$$

The first bound is the original Strichartz estimate (see [48]) and the second follows from scaling. These will be augmented with certain lateral Strichartz estimates to be introduced shortly. Strichartz estimates alone are not sufficient for controlling the nonlinearity in (2.7). The additional control required comes from local smoothing and maximal function estimates. Certain local smoothing spaces localized to cubes were introduced in [26] to study the local wellposedness of Schrödinger equations with general derivative nonlinearities. Stronger spaces were introduced in [20] to prove a low-regularity global result. In the Schrödinger map setting, local smoothing spaces were first used in [19] and subsequently in [21, 2, 6]. The particular local smoothing/maximal function spaces we shall use were introduced in [4].

Define the lateral spaces $L^{p,q}_\theta$ as those consisting of all measurable f for which the norm

$$\|f\|_{L^{p,q}_\theta} = \left(\int_{x \cdot \theta^\perp = 0} \left(\int_{\mathbf{R}} \int_{x \cdot \theta = 0} |f(x,t)|^q dx dt \right)^{p/q} dx \right)^{1/p}$$

is finite. Here, as context suggests, the inner "dx" denotes 1-dimensional Lebesgue measure on the line given by $\{x \cdot \theta = 0\}$, and the outer "dx" denotes Lebesgue measure on the orthogonal line through the origin. Also, $\theta^\perp := i\theta$. We make the usual modifications when $p = \infty$ or $q = \infty$. The most important spaces for our analysis are the local smoothing spaces $L^{\infty,2}_\theta$ and the inhomogeneous local smoothing spaces $L^{1,2}_\theta$. To move between these spaces we use the maximal function spaces $L^{2,\infty}_\theta$.

The following two estimates were shown in [19] and [21]:

Lemma 3.3 (Local smoothing)**.** *Let* $f \in L^2_x(\mathbf{R}^2)$*,* $k \in \mathbf{Z}$*, and* $\theta \in \mathbf{S}^1$*. Then*

$$\|e^{it\Delta} P_{k,\theta} f\|_{L^{\infty,2}_\theta} \lesssim 2^{-k/2} \|f\|_{L^2_x}.$$

For $f \in L^2_x(\mathbf{R}^d)$*, the maximal function space bound*

$$\|e^{it\Delta} P_k f\|_{L^{2,\infty}_\theta} \lesssim 2^{k(d-1)/2} \|f\|_{L^2_x}$$

holds for dimension $d \geq 3$*.*

In $d = 2$, the maximal function bound fails due to a logarithmic divergence. In order to overcome this, we exploit Galilean invariance as in [4] (the idea goes back to [60] in the setting of wave maps).

For $p, q \in [1, \infty]$, $\theta \in \mathbf{S}^1$, $\lambda \in \mathbf{R}$, define $L^{p,q}_{\theta,\lambda}$ using the norm

$$\|f\|_{L^{p,q}_{\theta,\lambda}} = \|T_{\lambda\theta}(f)\|_{L^{p,q}_\theta} = \left(\int_{x\cdot\theta^\perp=0} \left(\int_{\mathbf{R}} \int_{x\cdot\theta=0} |f(x + t\lambda\theta, t)|^q dxdt \right)^{p/q} dx \right)^{1/p},$$

where T_w denotes the Galilean transformation

$$T_w(f)(x, t) = e^{-ix\cdot w/2} e^{-it|w|^2/4} f(x + tw, t).$$

With $W \subset \mathbf{R}$ finite we define the spaces $L^{p,q}_{\theta,W}$ by

$$L^{p,q}_{\theta,W} = \sum_{\lambda\in W} L^{p,q}_{\theta,\lambda}, \qquad \|f\|_{L^{p,q}_{\theta,W}} = \inf_{f=\sum_{\lambda\in W} f_\lambda} \sum_{\lambda\in W} \|f_\lambda\|_{L^{p,q}_{\theta,\lambda}}.$$

For $k \in \mathbf{Z}$, $\mathcal{K} \in \mathbf{Z}_+$, set

$$W_k := \{\lambda \in [-2^k, 2^k] : 2^{k+2\mathcal{K}}\lambda \in \mathbf{Z}\}.$$

In our application we shall work on a finite time interval $[-2^{2\mathcal{K}}, 2^{2\mathcal{K}}]$ in order to ensure that the W_k are finite. This still suffices for proving global results so long as our effective bounds are proved with constants independent of $T, \mathcal{K}$. As discussed in [4, §3], restricting T to a finite time interval avoids introducing additional technicalities.

Lemma 3.4 (Local smoothing/maximal function estimates)**.** *Let $f \in L^2_x(\mathbf{R}^2)$, $k \in \mathbf{Z}$, and $\theta \in \mathbf{S}^1$. Then*

$$\|e^{it\Delta} P_{k,\theta} f\|_{L^{\infty,2}_{\theta,\lambda}} \lesssim 2^{-k/2} \|f\|_{L^2_x}, \qquad |\lambda| \leq 2^{k-40},$$

and moreover, if $T \in (0, 2^{2\mathcal{K}}]$, then

$$\|1_{[-T,T]}(t) e^{it\Delta} P_k f\|_{L^{2,\infty}_{\theta,W_{k+40}}} \lesssim 2^{k/2} \|f\|_{L^2_x}.$$

Proof. The first bound follows from Lemma 3.3 via a Galilean boost. The second is more involved and proven in [4, §7]. □

Lemma 3.5 (Lateral Strichartz estimates)**.** *Let $f \in L^2_x(\mathbf{R}^2)$, $k \in \mathbf{Z}$, and $\theta \in \mathbf{S}^1$. Let $2 < p \leq \infty, 2 \leq q \leq \infty$ and $1/p + 1/q = 1/2$. Then*

$$\|e^{it\Delta} P_{k,\theta} f\|_{L^{p,q}_\theta} \lesssim 2^{k(2/p-1/2)} \|f\|_{L^2_x}, \qquad p \geq q,$$

$$\|e^{it\Delta} P_k f\|_{L^{p,q}_\theta} \lesssim_p 2^{k(2/p-1/2)} \|f\|_{L^2_x}, \qquad p \leq q.$$

Proof. Informally speaking, these bounds follow from interpolating between the L^4 Strichartz estimate and the local smoothing/maximal function estimates of Lemma 3.4. See [4, Lemma 7.1] for the rigorous argument. □

We now introduce the main function spaces. Let $T > 0$. For $k \in \mathbf{Z}$, let $I_k = \{\xi \in \mathbf{R}^2 : |\xi| \in [2^{k-1}, 2^{k+1}]\}$. Let

$$L^2_k(T) := \{f \in L^2(\mathbf{R}^2 \times [-T,T]) : \operatorname{supp} \hat{f}(\xi,t) \subset I_k \times [-T,T]\}.$$

For $f \in L^2(\mathbf{R}^2 \times [-T,T])$, let

$$\begin{aligned}\|f\|_{F^0_k(T)} &:= \|f\|_{L^\infty_t L^2_x} + \|f\|_{L^4_{t,x}} + 2^{-k/2}\|f\|_{L^4_x L^\infty_t} \\ &\quad + 2^{-k/6} \sup_{\theta \in \mathbf{S}^1} \|f\|_{L^{3,6}_\theta} + +2^{-k/2} \sup_{\theta\in\mathbf{S}^1} \|f\|_{L^{2,\infty}_{\theta,W_{k+40}}}.\end{aligned}$$

We then define, as in [4], $F_k(T)$, $G_k(T)$, $N_k(T)$ as the normed spaces of functions in $L^2_k(T)$ for which the corresponding norms are finite:

$$\begin{aligned}\|f\|_{F_k(T)} &:= \inf_{J, m_1,\dots,m_J \in \mathbf{Z}_+} \inf_{f = f_{m_1} + \cdots + f_{m_J}} \sum_{j=1}^J 2^{m_j} \|f_{m_j}\|_{F^0_{k+m_j}} \\ \|f\|_{G_k(T)} &:= \|f\|_{F^0_k(T)} + 2^{k/6} \sup_{|j-k|\le 20} \sup_{\theta\in\mathbf{S}^1} \|P_{j,\theta} f\|_{L^{6,3}_\theta} \\ &\quad + 2^{k/2} \sup_{|j-k|\le 20} \sup_{\theta\in\mathbf{S}^1} \sup_{|\lambda| < 2^{k-40}} \|P_{j,\theta} f\|_{L^{\infty,2}_{\theta,\lambda}} \\ \|f\|_{N_k(T)} &:= \inf_{f = f_1+f_2+f_3+f_4+f_5+f_6} \|f_1\|_{L^{4/3}_{t,x}} + 2^{k/6}\|f_2\|_{L^{3/2,6/5}_{\hat\theta_1}} + 2^{k/6}\|f_3\|_{L^{3/2,6/5}_{\hat\theta_2}} \\ &\quad + 2^{-k/6}\|f_4\|_{L^{6/5,3/2}_{\hat\theta_1}} + 2^{-k/6}\|f_5\|_{L^{6/5,3/2}_{\hat\theta_2}} + 2^{-k/2} \sup_{\theta\in\mathbf{S}^1} \|f_6\|_{L^{1,2}_{\theta,W_{k-40}}},\end{aligned}$$

where $(\hat\theta_1, \hat\theta_2)$ denotes the canonical basis in $\mathbf{R}^2$.

There are a few minor differences between these spaces and those appearing in [4]. The space F^0_k now includes the angular Strichartz space $L^{3,6}_\theta$, whereas in [4], only G_k was endowed with this norm. The net effect on the space G_k is that it is left unchanged. The space F_k, however, now explicitly incorporates this particular angular Strichartz structure. Note though, that for fixed $\theta \in \mathbf{S}^1$, we have by enough applications of Young's and Hölder's inequalities that

$$\begin{aligned}2^{-k/6}\|f\|_{L^{3,6}_\theta} &= 2^{-k/6} \left(\int_{x\cdot\theta^\perp = 0} \left(\int_{\mathbf{R}} \int_{x\cdot\theta=0} |f(x_1\theta + x_2, t)|^6 dx_2 dt \right)^{1/2} dx_1 \right)^{1/3} \\ &\lesssim 2^{-k/6} \left(\int_{x\cdot\theta^\perp=0} \|f\|^2_{L^4_{\theta,t}} \|f\|_{L^\infty_{\theta,t}} dx_1 \right)^{1/3} \\ &\lesssim 2^{-k/6} \left(\int_{x\cdot\theta^\perp=0} \|f\|^4_{L^4_{\theta,t}} dx_1 \right)^{1/6} \left(\int_{x\cdot\theta^\perp=0} \|f\|^2_{L^\infty_{\theta,t}} dx_1 \right)^{1/6} \\ &\lesssim \|f\|^{2/3}_{L^4} \cdot 2^{-k/6} \|f\|^{1/3}_{L^{2,\infty}_\theta} \\ &\lesssim \|f\|_{L^4} + 2^{-k/2}\|f\|_{L^{2,\infty}_\theta}.\end{aligned}$$

We also make one change to the N_k space: We explicitly incorporate $L_\theta^{6/5,3/2}$.

Incorporating these extra angular Strichartz spaces affords us greater flexibility in certain estimates: We can avoid having to use local smoothing/maximal function spaces if we are willing to give up some decay. This tradeoff pays off in §5, where as a consequence we can prove a stronger local smoothing estimate for a certain magnetic nonlinear Schrödinger equation in the one regime where this improvement is absolutely essential.

Proposition 3.6 (Main linear estimate)**.** *Assume $\mathcal{K} \in \mathbf{Z}_+$, $T \in (0, 2^{2\mathcal{K}}]$ and $k \in \mathbf{Z}$. Then for each $u_0 \in L^2$ that is frequency-localized to I_k and for any $h \in N_k(T)$, the solution u of*

$$(i\partial_t + \Delta_x)u = h, \qquad u(0) = u_0$$

satisfies

$$\|u\|_{G_k(T)} \lesssim \|u(0)\|_{L^2_x} + \|h\|_{N_k(T)}.$$

Proof. See [4, Proposition 7.2] for details. Our changes to the spaces necessitate only minor changes in their proof, as we must incorporate $L_{\hat{\theta}_1}^{6/5,3/2} and L_{\hat{\theta}_2}^{6/5,3/2}$ into the space $N_k^0(T)$. □

The spaces $G_k(T)$ are used to hold projections $P_k\psi_m$ of the derivative fields ψ_m satisfying (2.7). The main components of $G_k(T)$ are the local smoothing/maximal function spaces $L^{\infty,2}_{\theta,\lambda}$, $L^{2,\infty}_{\theta,W_{k+40}}$, and the angular Strichartz spaces. The local smoothing and maximal function space components play an essential role in recovering the derivative loss that is due to the magnetic nonlinearity.

The spaces $N_k(T)$ hold frequency projections of the nonlinearities in (2.7). Here the main spaces are the inhomogeneous local smoothing spaces $L^{1,2}_{\theta,W_{k-40}}$ and the Strichartz spaces, both chosen to match those of $G_k(T)$.

The spaces $G_k(T)$ clearly embed in $F_k(T)$. Two key properties enjoyed only by the larger spaces $F_k(T)$ are

$$\|f\|_{F_k(T)} \approx \|f\|_{F_{k+1}(T)},$$

for $k \in \mathbf{Z}$ and $f \in F_k(T) \cap F_{k+1}(T)$, and

$$\|P_k(uv)\|_{F_k(T)} \lesssim \|u\|_{F_{k'}(T)} \|v\|_{L^\infty_{t,x}}$$

for $k, k' \in \mathbf{Z}$, $|k - k'| \leq 20$, $u \in F_{k'}(T)$, $v \in L^\infty(\mathbf{R}^2 \times [-T, T])$. Both of these properties follow readily from the definitions.

In order to bound the nonlinearity of (2.7) in $N_k(T)$, it is important to gain regularity from the parabolic heat-time smoothing effect. The desired frequency-localized bounds do not (or at least not so readily) propagate in

heat-time in the spaces $G_k(T)$, whereas these bounds do propagate with decay in the larger spaces $F_k(T)$. Note that since the $F_k(T)$ norm is translation invariant, it holds that

$$\|e^{s\Delta}h\|_{F_k(T)} \lesssim (1+s2^{2k})^{-20}\|h\|_{F_k(T)} \qquad s \geq 0,$$

for $h \in F_k(T)$. In certain bilinear estimates we do not need the full strength of the spaces $F_k(T)$ and instead can use the bound

$$\|f\|_{F_k(T)} \lesssim \|f\|_{L_x^2 L_t^\infty} + \|f\|_{L_{t,x}^4}, \tag{3.1}$$

which follows from

$$\|f\|_{L^{2,\infty}_{\theta, W_{k+m_j}}} \leq \|f\|_{L_\theta^{2,\infty}} \lesssim 2^{k/2}\|f\|_{L_x^2 L_t^\infty}.$$

We introduce one more class of function spaces. These can be viewed as a refinement of the Strichartz part of $F_k(T)$. For $k \in \mathbf{Z}$ and $\omega \in [0,1/2]$ we define $S_k^\omega(T)$ to be the normed space of functions belonging to $L_k^2(T)$ whose norm

$$\|f\|_{S_k^\omega(T)} = 2^{\omega k}\left(\|f\|_{L_t^\infty L_x^{2_\omega}} + \|f\|_{L_t^4 L_x^{p_\omega}} + 2^{-k/2}\|f\|_{L_x^{p_\omega} L_t^\infty}\right) \tag{3.2}$$

is finite, where the exponents 2_ω and p_ω are determined by

$$\frac{1}{2_\omega} - \frac{1}{2} = \frac{1}{p_\omega} - \frac{1}{4} = \frac{\omega}{2}.$$

Note that $F_k(T) \hookrightarrow S_k^0(T)$ and that by Bernstein we have

$$\|f\|_{S_k^{\omega'}(T)} \lesssim \|f\|_{S_k^\omega(T)}, \qquad \omega' \leq \omega.$$

3.2. **Bilinear estimates.**

Lemma 3.7 (Bilinear estimates on $N_k(T)$)**.** *For $k, k_1, k_3 \in \mathbf{Z}$, $h \in L_{t,x}^2$, $f \in F_{k_1}(T)$, and $g \in G_{k_3}(T)$, we have the following inequalities under the given restrictions on k_1, k_3.*

$$|k_1 - k| \leq 80: \quad \|P_k(hf)\|_{N_k(T)} \lesssim \|h\|_{L_{t,x}^2}\|f\|_{F_{k_1}(T)} \tag{3.3}$$

$$k_1 \leq k - 80: \quad \|P_k(hf)\|_{N_k(T)} \lesssim 2^{-|k-k_1|/6}\|h\|_{L_{t,x}^2}\|f\|_{F_{k_1}(T)} \tag{3.4}$$

$$k \leq k_3 - 80: \quad \|P_k(hg)\|_{N_k(T)} \lesssim 2^{-|k-k_3|/6}\|h\|_{L_{t,x}^2}\|g\|_{G_{k_3}(T)}. \tag{3.5}$$

Proof. Estimate (3.3) follows from Hölder's inequality and the definition of $F_k(T), N_k(T)$:

$$\|Ff\|_{L^{4/3}} \leq \|F\|_{L^2}\|f\|_{L^4}.$$

For (3.4) and (3.5), we use an angular partition of unity in frequency to write

$$f = f_1 + f_2, \qquad \|f_1\|_{L_{\hat{\theta}_1}^{3,6}} + \|g_1\|_{L_{\hat{\theta}_2}^{3,6}} \lesssim 2^{k_1/6}\|f\|_{F_k(T)}.$$

and

$$g = g_1 + g_2, \qquad \|g_1\|_{L^{6,3}_{\hat\theta_1}} + \|g_1\|_{L^{6,3}_{\hat\theta_2}} \lesssim 2^{-k_1/6}\|g\|_{G_k(T)}.$$

Then

$$\begin{aligned}\|P_k(Ff)\|_{N_k(T)} &\lesssim 2^{-k/6}\left(\|Ff_1\|_{L^{6/5,3/2}_{\hat\theta_1}} + \|Ff_2\|_{L^{6/5,3/2}_{\hat\theta_2}}\right)\\ &\lesssim 2^{-k/6}\|F\|_{L^2}\left(\|f_1\|_{L^{3,6}_{\hat\theta_1}} + \|f_1\|_{L^{3,6}_{\hat\theta_2}}\right)\\ &\lesssim 2^{(k_1-k)/6}\|F\|_{L^2}\|f\|_{F_{k_1}(T)}.\end{aligned}$$

and

$$\begin{aligned}\|P_k(Fg)\|_{N_k(T)} &\lesssim 2^{k/6}\left(\|Fg_1\|_{L^{3/2,6/5}_{\hat\theta_1}} + \|Fg_2\|_{L^{3/2,6/5}_{\hat\theta_2}}\right)\\ &\lesssim 2^{k/6}\|F\|_{L^2}\left(\|g_1\|_{L^{6,3}_{\hat\theta_1}} + \|g_1\|_{L^{6,3}_{\hat\theta_2}}\right)\\ &\lesssim 2^{(k-k_1)/6}\|F\|_{L^2}\|g\|_{G_{k_3}(T)}.\end{aligned}$$

□

Lemma 3.8 (Bilinear estimates on $L^2_{t,x}$)**.** *For $k_1, k_2, k_3 \in \mathbf{Z}$, $f_1 \in F_{k_1}(T)$, $f_2 \in F_{k_2}(T)$, and $g \in G_{k_3}(T)$, we have*

$$\|f_1 \cdot f_2\|_{L^2_{t,x}} \lesssim \|f_1\|_{F_{k_1}(T)}\|f_2\|_{F_{k_2}(T)} \tag{3.6}$$

$$k_1 \le k_3: \quad \|f \cdot g\|_{L^2_{t,x}} \lesssim 2^{-|k_1-k_3|/6}\|f\|_{F_{k_1}(T)}\|g\|_{G_{k_3}(T)}. \tag{3.7}$$

Proof. It suffices to show

$$\|fg\|_{L^2} \lesssim \|f\|_{F^0_{k_1}(T)}\|g\|_{G_{k_2}(T)}, \qquad k_1 \ge k_2 - 100 \tag{3.8}$$

and

$$\|fg\|_{L^2} \lesssim 2^{(k_1-k_2)/6}\|f\|_{F^0_{k_1}(T)}\|g\|_{G_{k_2}(T)}, \qquad k_1 < k_2 - 100. \tag{3.9}$$

Estimate (3.8) follows from estimating each factor in L^4. For (3.9), we first observe that, using a smooth partition of unity in frequency space, we may assume that $\hat g$ is supported in these set

$$\left\{\xi : |\xi| \in [2^{k_2-1}, 2^{k_2+1}] \text{ and } \xi \cdot \theta_0 \ge 2^{k_2-5}\right\}$$

for some direction $\theta_0 \in \mathbf{S}^1$. Then

$$\|fg\|_{L^2} \lesssim \|f\|_{L^{3,6}_{\theta_0}}\|g\|_{L^{6,3}_{\theta_0}} \lesssim 2^{(k_1-k_2)/6}\|f\|_{F^0_{k_1}(T)}\|g\|_{G_{k_2}(T)}$$

□

We also have the following stronger estimates, which rely upon the local smoothing and maximal function spaces.

Lemma 3.9 (Bilinear estimates using local smoothing/maximal function bounds)**.** *For $k, k_1, k_2 \in \mathbf{Z}$, $h \in L^2_{t,x}$, $f \in F_{k_1}(T)$, $g \in G_{k_2}(T)$, we have the following inequalities under the given restrictions on k_1, k_2.*

$$k_1 \le k - 80: \quad \|P_k(hf)\|_{N_k(T)} \lesssim 2^{-|k-k_1|/2} \|h\|_{L^2_{t,x}} \|f\|_{F_{k_1}(T)} \tag{3.10}$$

$$k_1 \le k_2: \quad \|f \cdot g\|_{L^2_{t,x}} \lesssim 2^{-|k_1-k_2|/2} \|f\|_{F_{k_1}(T)} \|g\|_{G_{k_2}(T)}. \tag{3.11}$$

Proof. Estimate (3.10) follows from the definitions since

$$\|P_k(hf)\|_{N_k(T)} \lesssim 2^{-k/2} \sup_{\theta \in \mathbf{S}^1} \|hf\|_{L^{1,2}_{\theta, W_{k-40}}} \lesssim 2^{-k/2} \sup_{\theta \in \mathbf{S}^1} \|f\|_{L^{2,\infty}_{\theta, W_{k_1+40}}} \|h\|_{L^2_{t,x}}.$$

The proof of (3.11) parallels that of (3.7) and is omitted. $\square$

3.3. **Trilinear Estimates and Summation.** We combine the bilinear estimates to establish some trilinear estimates. As we do not control local smoothing norms along the heat flow, we will oftentimes be able to put only one term in a G_k space. Nonetheless, such estimates still exhibit good off-diagonal decay.

Define the sets $Z_1(k), Z_2(k), Z_3(k) \subset \mathbf{Z}^3$ as follows:

$$\begin{aligned} Z_1(k) :=& \{(k_1, k_2, k_3) \in \mathbf{Z}^3 : k_1, k_2 \le k - 40 \text{ and } |k_3 - k| \le 4\} \\ Z_2(k) :=& \{(k_1, k_2, k_3) \in \mathbf{Z}^3 : k, k_3 \le k_1 - 40 \text{ and } |k_2 - k_1| \le 45\} \\ Z_3(k) :=& \{(k_1, k_2, k_3) \in \mathbf{Z}^3 : k_3 \le k \text{ and } |k - \max\{k_1, k_2\}| \le 40 \\ & \quad \textbf{or } k_3 > k \text{ and } |k_3 - \max\{k_1, k_2\}| \le 40\} \end{aligned} \tag{3.12}$$

In our main trilinear estimate, we avoid using local smoothing / maximal function spaces.

Lemma 3.10 (Main trilinear estimate)**.** *Let C_{k,k_1,k_2,k_3} denote the best constant C in the estimate*

$$\|P_k(P_{k_1} f_1 P_{k_2} f_2 P_{k_3} g)\|_{N_k(T)} \lesssim C \|P_{k_1} f_1\|_{F_{k_1}(T)} \|P_{k_2} f_2\|_{F_{k_2}(T)} \|P_{k_3} g\|_{G_{k_3}(T)}. \tag{3.13}$$

The best constant C_{k,k_1,k_2,k_3} satisfies the bounds

$$C_{k,k_1,k_2,k_3} \lesssim \begin{cases} 2^{-|(k_1+k_2)/6 - k/3|} & (k_1, k_2, k_3) \in Z_1(k) \\ 2^{-|k-k_3|/6} & (k_1, k_2, k_3) \in Z_2(k) \\ 2^{-|\Delta k|/6} & (k_1, k_2, k_3) \in Z_3(k) \\ 0 & (k_1, k_2, k_3) \in \mathbf{Z}^3 \setminus \{Z_1(k) \cup Z_2(k) \cup Z_3(k)\}, \end{cases}$$

where $\Delta k = \max\{k, k_1, k_2, k_3\} - \min\{k, k_1, k_2, k_3\} \ge 0$.

Proof. After placing the term $P_k(P_{k_1} f_1 P_{k_2} f_2 P_{k_3} g)$ in $L^{4/3}_{t,x}$ and then using Hölder's inequality to bound each factor in $L^4_{t,x}$, it follows from Bernstein that

$$C_{k,k_1,k_2,k_3} \lesssim 1, \tag{3.14}$$

and so, in particular, for any choice of integers k, k_1, k_2, k_3, such a constant C_{k,k_1,k_2,k_3} exists.

Frequencies not represented in one of $Z_1(k), Z_2(k), Z_3(k)$ cannot interact so as to yield a frequency in I_k. Over $Z_1(k)$, we apply (3.4) and (3.7).

On $Z_2(k)$ we apply (3.4) if $k > k_3$ and (3.5) if $k \leq k_3$. We conclude with (3.6).

On $Z_3(k)$ we may assume without loss of generality that $k_1 \leq k_2$. First suppose that $k_3 \leq k$ and $|k - k_2| \leq 40$. If $k_1 \leq k_3$, then use (3.4), applying (3.6) to $P_{k_2} f_2 P_{k_3} g$. If $k_3 < k_1$, then use(3.6) on $P_{k_1} f_1 P_{k_2} f_2$ instead.

Now suppose that $k_3 > k$ and $|k_3 - k_2| \leq 40$. If $k_1 \leq k$, then use (3.3), applying (3.7) to $P_{k_1} f_1 P_{k_3} g$. If $k_{\min} = k$, then use (3.5) and (3.6). $\square$

Corollary 3.11. *Let $\{a_k\}$, $\{b_k\}$, $\{c_k\}$ be δ-frequency envelopes. Let C_{k,k_1,k_2,k_3} be as in Lemma 3.10. Then*

$$\sum_{(k_1,k_2,k_3) \in \mathbf{Z}^3 \setminus Z_2(k)} C_{k,k_1,k_2,k_3} a_{k_1} b_{k_2} c_{k_3} \lesssim a_k b_k c_k.$$

Proof. By Lemma 3.10, it suffices to restrict the sum to (k_1, k_2, k_3) lying in $Z_1(k) \cup Z_3(k)$. On $Z_1(k)$, the sum is bounded by

$$\begin{aligned}
&\sum_{(k_1,k_2,k_3) \in Z_1(k)} 2^{-|(k_1+k_2)/6 - k/3|} a_{k_1} b_{k_2} c_{k_3} \\
&\lesssim \sum_{k_1, k_2 \leq k-40} 2^{-|(k_1+k_2)/6 - k/3|} 2^{\delta |2k - k_1 - k_2|} a_k b_k c_k \\
&\lesssim a_k b_k c_k.
\end{aligned}$$

On Z_3, we may assume without loss of generality that $k_2 \leq k_1$. The sum is then controlled by

$$\begin{aligned}
&\sum_{(k_1,k_2,k_3) \in Z_3(k)} 2^{-|\Delta k|/6} a_{k_1} b_{k_2} c_{k_3} \\
&\lesssim \sum_{\substack{k_2 \leq k \\ k_3 \leq k \\ |k_1 - k| \leq 40}} 2^{-|k - \min\{k_2, k_3\}|/6} a_{k_1} b_{k_2} c_{k_3} + \sum_{\substack{k_2 \leq k_1 \\ k_1 > k \\ |k_3 - k_1| \leq 40}} 2^{-|k_1 - \min\{k_2, k\}|/6} a_{k_1} b_{k_2} c_{k_3} \\
&\lesssim \sum_{\substack{k_2 \leq k \\ k_3 \leq k}} 2^{-|k - \min\{k_2, k_3\}|/6} a_k b_{k_2} c_{k_3} + \sum_{\substack{k_2 \leq k_1 \\ k_1 > k}} 2^{-|k_1 - \min\{k_2, k\}|/6} a_{k_1} b_{k_2} c_{k_1}.
\end{aligned}$$

The first of these summands is controlled by

$$\begin{aligned}
&\sum_{k_3\le k_2\le k} 2^{-|k-k_3|/6} a_k b_{k_2} c_{k_3} + \sum_{k_2<k_3\le k} 2^{-|k-k_2|/6} a_k b_{k_2} c_{k_3} \\
&\lesssim \sum_{k_3\le k_2\le k} 2^{-|k-k_3|/6} 2^{\delta|k-k_2|} a_k b_k c_{k_3} + \sum_{k_2<k_3\le k} 2^{-|k-k_2|/6} 2^{\delta|k-k_3|} a_k b_{k_2} c_k \\
&\lesssim \sum_{k_3\le k} 2^{(\delta-1/6)|k-k_3|} a_k b_k c_{k_3} + \sum_{k_2<k} 2^{(\delta-1/6)|k-k_2|} a_k b_{k_2} c_k \\
&\lesssim \sum_{k_3\le k} 2^{(2\delta-1/6)|k-k_3|} a_k b_k c_k + \sum_{k_2<k} 2^{(2\delta-1/6)|k-k_2|} a_k b_k c_k \\
&\lesssim a_k b_k c_k.
\end{aligned}$$

The second is controlled by

$$\begin{aligned}
&\sum_{k\le k_2\le k_1} 2^{-|k_1-k|/6} a_{k_1} b_{k_2} c_{k_1} + \sum_{k_2<k\le k_1} 2^{-|k_1-k_2|/6} a_{k_1} b_{k_2} c_{k_1} \\
&\lesssim \sum_{k\le k_2\le k_1} 2^{-|k_1-k|/6} 2^{\delta|k_2-k|} a_{k_1} b_k c_{k_1} + \sum_{k_2<k\le k_1} 2^{|k_1-k_2|/6} 2^{\delta|k_2-k|} a_{k_1} b_k c_{k_1} \\
&\lesssim \sum_{k\le k_1} 2^{(\delta-1/6)|k_1-k|} a_{k_1} b_k c_{k_1} + \sum_{k_2<k\le k_1} 2^{(\delta-1/6)|k_1-k_2|} a_{k_1} b_k c_{k_1} \\
&\lesssim \sum_{k\le k_1} 2^{(3\delta-1/6)|k_1-k|} a_k b_k c_k + \sum_{k_2<k\le k_1} 2^{(3\delta-1/6)|k_1-k_2|} a_k b_k c_k \\
&\lesssim a_k b_k c_k.
\end{aligned}$$

□

Corollary 3.12. *Let $\{a_k\},\{b_k\}$ be δ-frequency envelopes. Let C_{k,k_1,k_2,k_3} be as in Lemma 3.10. Then*

$$\sum_{(k_1,k_2,k_3)\in Z_2(k)\cup Z_3(k)} 2^{\max\{k,k_3\}-\max\{k_1,k_2\}} C_{k,k_1,k_2,k_3} a_{k_1} b_{k_2} c_{k_3} \lesssim a_k b_k c_k$$

Proof. On $Z_3(k)$, $\max\{k_1,k_2\}\sim\max\{k,k_3\}$, and so the bound on $Z_3(k)$ follows from Corollary 3.11.

Note that $\max\{k_1,k_2\}>\max\{k,k_3\}$ on Z_2, where the sum is controlled by

$$\begin{aligned}
&\sum_{(k_1,k_2,k_3)\in Z_2(k)} 2^{\max\{k,k_3\}-\max\{k_1,k_2\}} 2^{-|k-k_3|/6} a_{k_1} b_{k_2} c_{k_3} \\
&\lesssim \sum_{k,k_3\le k_1-40} 2^{\max\{k,k_3\}-k_1} 2^{-|k-k_3|/6} a_{k_1} b_{k_1} c_{k_3},
\end{aligned}$$

Restricting the sum to $k_3\le k$, we get

$$\sum_{k_3\le k\le k_1-40} 2^{-|k-k_1|} 2^{-|k-k_3|/6} a_{k_1} b_{k_1} c_{k_3} \lesssim a_k b_k c_k$$

Over the complementary range $k \le k_3 \le k_1 - 40$, we have

$$\begin{aligned}&\sum_{k\le k_3\le k_1-40} 2^{-|k_3-k_1|}2^{-|k-k_3|/6}a_{k_1}b_{k_1}c_{k_3}\\ &\lesssim a_k b_k c_k \sum_{k\le k_3\le k_1-40} 2^{-|k_3-k_1|}2^{-|k-k_3|/6}2^{2\delta|k_1-k|}2^{\delta|k-k-3|}.\end{aligned}$$

Performing the change of variables $j := k_1 - k_3, \ell := k_3 - k$, we control the sum by

$$\sum_{j,\ell\ge 0} 2^{-j}2^{-\ell/6}2^{2\delta(j+\ell)}2^{\delta\ell} \lesssim \sum_{j,\ell\ge 0} 2^{(2\delta-1)j}2^{(3\delta-1/6)\ell} \lesssim 1$$

□

Taking advantage of the local smoothing/maximal function spaces, we can obtain the following improvement.

Lemma 3.13 (Main trilinear estimate improvement over Z_1)**.** *The best constant C_{k,k_1,k_2,k_3} in* (3.13) *satisfies the improved estimate*

$$C_{k,k_1,k_2,k_3} \lesssim 2^{-|(k_1+k_2)/2-k|} \tag{3.15}$$

when $\{k_1, k_2, k_3\} \in Z_1(k)$.

There are certain situations, such as when bounding the cubic-type term, where we can place each term in a G_k space.

Lemma 3.14 (Improved trilinear estimate)**.** *Let C_{k,k_1,k_2,k_3} denote the best constant C in the estimate*

$$\|P_k\left(P_{k_1}g_1P_{k_2}g_2P_{k_3}g_3\right)\|_{N_k(T)} \lesssim C\|P_{k_1}g_1\|_{G_{k_1}(T)}\|P_{k_2}g_2\|_{G_{k_2}(T)}\|P_{k_3}g_3\|_{G_{k_3}(T)}. \tag{3.16}$$

The best constant C_{k,k_1,k_2,k_3} satisfies the bounds

$$C_{k,k_1,k_2,k_3} \lesssim \begin{cases} 2^{-|(k_1+k_2)/2-k|} & (k_1,k_2,k_3)\in Z_1(k)\\ 2^{-|k-k_1|/6}2^{-|k_3-k_2|/6} & (k_1,k_2,k_3)\in Z_2(k)\\ 2^{-|\Delta k|/6} & (k_1,k_2,k_3)\in Z_3(k)\\ 0 & (k_1,k_2,k_3)\in \mathbf{Z}^3\setminus\{Z_1(k)\cup Z_2(k)\cup Z_3(k)\},\end{cases}$$

where $\Delta k = \max\{k,k_1,k_2,k_3\} - \min\{k,k_1,k_2,k_3\} \ge 0$.

Proof. We seek an improvement over Lemma 3.10 only on the set $Z_2(k)$. Here we apply (3.5) so that

$$\|P_k(P_{k_1}g_1P_{k_2}g_2P_{k_3}g_3)\|_{N_k(T)} \lesssim 2^{-|k-k_1|/6}\|P_{k_1}g_1P_{k_2}g_2\|_{L^2_{t,x}}\|P_{k_3}g_3\|_{G_{k_3}(T)}.$$

We conclude with (3.7). □

Corollary 3.15. *Let $\{a_k\}$, $\{b_k\}$, $\{c_k\}$ be δ-frequency envelopes. Let C_{k,k_1,k_2,k_3} be as in Lemma 3.14. Then*

$$\sum_{(k_1,k_2,k_3)\in\mathbf{Z}^3} C_{k,k_1,k_2,k_3} \lesssim a_k b_k c_k$$

Proof. In view of Corollary (3.11) we need only establish the bound on $Z_2(k)$. We have

$$\begin{aligned}
&\sum_{(k_1,k_2,k_3)\in Z_2(k)} 2^{-|k-k_1|/6} 2^{-|k_3-k_2|/6} a_{k_1} b_{k_2} c_{k_3} \\
&\lesssim \sum_{k,k_3\leq k_1-40} 2^{-|k-k_1|/6} 2^{-|k_3-k_1|/6} a_{k_1} b_{k_1} c_{k_3} \\
&\lesssim a_k b_k c_k \sum_{k,k_3\leq k_1-40} 2^{-|k-k_1|/6} 2^{-|k_3-k_1|/6} 2^{3\delta|k-k_1|} 2^{\delta|k_3-k_1|}.
\end{aligned}$$

To finish, sum on k_3, then on k_1. □

4. Perturbative reductions

One strategy for proving (1.15), i.e., $b_k \lesssim c_k$ (see (1.7) and (1.11) for definitions of the b_k and c_k frequency envelopes), is to show that the nonlinearity is perturbative. Toward this end, we may project (2.7) to frequencies $\sim 2^k$ using the Littlewood-Paley multiplier P_k and then apply the linear estimate of Proposition 3.6:

$$\|P_k \psi_m\|_{G_k(T)} \lesssim \|P_k \psi_m(0)\|_{L_x^2} + \|P_k \mathcal{N}_m\|_{N_k(T)}. \tag{4.1}$$

If the nonlinearity is perturbative, i.e., $\|P_k \mathcal{N}_m\|_{N_k(T)} \lesssim \varepsilon b_k$, then (4.1) implies $b_k \lesssim c_k + \varepsilon b_k$, which establishes (1.15). This is the general approach of [4] in the case of small critical norm. In the dispersed setting, $P_k \mathcal{N}_m$ cannot be shown to obey such a bound in a straightforward way; it fails to be perturbative in this sense. Nevertheless several pieces of $\mathcal{N}_m$ can in fact be shown to be perturbative, and moreover, doing so plays an important role in enabling us to gain control over the remaining nonperturbative part of the nonlinearity.

Recall that we decompose $\mathcal{N}_m$ as $\mathcal{N}_m = B_m + V_m$, with B_m, V_m respectively given by (2.8), (2.9). Below we shall prove that V_m is perturbative. The magnetic term B_m we shall further decompose, showing that part of it is also perturbative. As we shall frequently represent derivative fields and connection coefficients in terms of integrals along the heat flow, we first recall and establish several estimates along this flow.

Throughout this section, $\varepsilon > 0$ is a very small number such that $b_k \leq \varepsilon$ and $\varepsilon^{1/2} \sum_j b_j^2 \ll 1$. In application, it will be set equal to a suitable power of the energy dispersion parameter ε_0.

4.1. **Bounds along the heat flow.** We recall from [4, 43] some estimates in F_k and $S_k^{1/2}$ spaces that propagate along the heat flow.

Lemma 4.1. *Let $k \in \mathbf{Z}$, $s \geq 0$. Let $F \in \{A_\ell^2, \partial_\ell A_\ell, fg : \ell = 1, 2; f, g \in \{\psi_m, \overline{\psi_m} : m = 1, 2\}\}$. Then*

$$\|P_k\psi_m(s)\|_{F_k(T)\cap S_k^{1/2}(T)} \lesssim (1 + s2^{2k})^{-4}2^{-\sigma k}b_k(\sigma) \tag{4.2}$$

$$\|P_k(A_\ell\psi_m(s))\|_{F_k(T)\cap S_k^{1/2}(T)} \lesssim \varepsilon(1 + s2^{2k})^{-3}(s2^{2k})^{-3/8}2^k2^{-\sigma k}b_k(\sigma) \tag{4.3}$$

$$\|P_kF(s)\|_{F_k(T)\cap S_k^{1/2}(T)} \lesssim \varepsilon^{1/2}(1 + s2^{2k})^{-2}(s2^{2k})^{-5/8}2^k2^{-\sigma k}b_k(\sigma) \tag{4.4}$$

$$\|P_k \int_0^s e^{(s-s')\Delta}U_m(s')ds'\|_{F_k(T)} \lesssim \varepsilon(1 + s2^{2k})^{-4}2^{-\sigma k}b_k(\sigma). \tag{4.5}$$

We also have

$$\|P_kA_m(0)\|_{F_k(T)\cap S_k^{1/2}(T)} \lesssim \sum_p b_p^2. \tag{4.6}$$

Using these $F_k \cap S_k^{1/2}$ estimates, we now establish several L^4 bounds along the heat flow.

Lemma 4.2. *Assume that $T \in (0, 2^{2\mathcal{K}}]$, $f, g \in H^{\infty,\infty}(T)$, $P_kf \in S_k^\omega(T)$, and $P_kg \in L^4_{t,x}$ for some $\omega \in [0, 1/2]$ and all $k \in \mathbf{Z}$. Set*

$$\mu_k := \sum_{|j-k|\leq 20} \|P_jf\|_{S^\omega_{k'}(T)}, \qquad \nu_k := \sum_{|j-k|\leq 20} \|P_jg\|_{L^4_{t,x}}.$$

Then, for any $k \in \mathbf{Z}$,

$$\|P_k(fg)\|_{L^4_{t,x}} \lesssim \sum_{j\leq k} 2^j\mu_j\nu_k + \sum_{j\leq k} 2^{(k+j)/2}\mu_k\nu_j + 2^k \sum_{j\geq k} 2^{-\omega(j-k)}\mu_j\nu_j.$$

Proof. For the proof, see [4, §5]. □

Lemma 4.3.

$$\|P_k\psi_s(0)\|_{L^4_{t,x}} \lesssim 2^k(1 + \sum_j b_j^2)b_k \tag{4.7}$$

$$\|P_k\psi_t(0)\|_{L^4_{t,x}} \lesssim 2^k(1 + \sum_j b_j^2)b_k \tag{4.8}$$

Proof. As $\psi_t(s = 0) = i\psi_s(s = 0)$, it suffices to prove (4.7). Recalling $\psi_s(0) = D_\ell(0)\psi_\ell(0)$, we have

$$\psi_s(0) = \partial_\ell\psi_\ell(0) - iA_\ell(0)\psi_\ell(0).$$

Clearly

$$\|P_k \partial_\ell \psi_\ell(0)\|_{L^4_{t,x}} \lesssim 2^k \|P_k \psi_x(0)\|_{L^4_{t,x}} \lesssim 2^k b_k.$$

For the remaining term, we apply Lemma 4.2, using (4.6) to bound $P_j A_\ell(0)$ in $S_j^{1/2}$ by $\sum_p b_p^2$. We get

$$\|P_k(A_\ell(0)\psi_\ell(0))\|_{L^4_{t,x}} \lesssim \sum_{j \le k} 2^j (\sum_p b_p^2) b_k + \sum_{j \le k} 2^{(k+j)/2} (\sum_p b_p^2) b_j + \\ 2^k \sum_{j \ge k} 2^{-(j-k)/2} (\sum_p b_p^2) b_j.$$

Therefore

$$\|P_k(A_\ell \psi_\ell(0))\|_{L^4_{t,x}} \lesssim 2^k (\sum_j b_j^2) b_k.$$

□

Corollary 4.4. *It holds that*

$$\|P_k \psi_t(s)\|_{L^4_{t,x}} + \|P_k \psi_s(s)\|_{L^4_{t,x}} \lesssim (1 + s 2^{2k})^{-2} 2^k (1 + \sum_j b_j^2) b_k.$$

Proof. The derivative fields ψ_α, $\alpha = 0, 3$, admit the Duhamel representations

$$\psi_\alpha(s) = e^{s\Delta} \psi_\alpha(0) + \int_0^s e^{(s-r)\Delta} U_\alpha(r)\, dr.$$

We claim that

$$\left\| \int_0^s e^{(s-r)\Delta} P_k U_\alpha(r)\, dr \right\|_{L^4_{t,x}} \lesssim \varepsilon (1 + s 2^{2k})^{-2} 2^k (1 + \sum_j b_j^2) b_k, \tag{4.9}$$

which combined with Lemma 4.3 and a standard iteration argument proves the lemma.

Let

$$F \in \{A_\ell^2, \partial_\ell A_\ell, fg : \ell = 1, 2; f, g \in \{\psi_m, \overline{\psi_m} : m = 1, 2\}\}.$$

By (4.4) we have

$$\|P_k F(r)\|_{S_k^{1/2}(T)} \lesssim \varepsilon^{1/2} (1 + r 2^{2k})^{-2} (r 2^{2k})^{-5/8} 2^k b_k.$$

Moreover, by (4.3)

$$\|P_k A_\ell(r)\|_{S_k^{1/2}(T)} \lesssim \varepsilon^{1/2} (1 + r 2^{2k})^{-3} (r 2^{2k})^{-1/8} b_k.$$

Applying Lemma 4.2 with $\omega = 1/2$ yields

$$\|P_k(F(r)\psi_\alpha(r))\|_{L^4_{t,x}} + 2^k \|P_k(A_\ell(r)\psi_\alpha(r)\|_{L^4_{t,x}} \lesssim \varepsilon (1 + r 2^{2k})^{-2} (r 2^{2k})^{-7/8} 2^k (1 + \sum_j b_j^2) b_k. \tag{4.10}$$

Integrating with respect to r yields

$$\int_0^s (1+(s-r)2^{2k})^{-N}(1+r2^{2k})^{-2}(r2^{2k})^{-7/8}dr \lesssim 2^{-2k}(1+s2^{2k})^{-2},$$

which, together with (4.10), implies (4.9). □

Corollary 4.5. *Let $\Psi = \psi_t - i\psi_s$. Then*

$$\|P_k\Psi\|_{L^4_{t,x}} \lesssim \varepsilon 2^k(1+\sum_j b_j^2)b_k.$$

Proof. From (2.21) and (2.22) we have

$$(\partial_s - \Delta)\Psi = U_t - iU_s.$$

As $\Psi(s=0)=0$, Duhamel implies

$$\Psi(s) = \int_0^s e^{(s-r)\Delta}(U_t - iU_s)(r)dr.$$

The conclusion follows from (4.9). □

Lemma 4.6. *It holds that*

$$\|P_kA_m(0)\|_{L^4_{t,x}} \lesssim b_k^2. \tag{4.11}$$

Proof. We have

$$\|P_k\psi_m(s)\|_{S_k^0} \lesssim (1+s2^{2k})^{-4}b_k$$

and

$$\|P_k(D_\ell\psi_\ell)(s)\|_{L^4_{t,x}} \lesssim (1+s2^{2k})^{-3}(s2^{2k})^{-3/8}2^kb_k.$$

Applying Lemma 4.2 with $\omega = 0$, we get

$$\begin{aligned}
\|P_kA_m(0)\|_{L^4_{t,x}} \lesssim &\int_0^\infty \|P_k(\overline{\psi_m(s)}D_\ell\psi_\ell(s))\|_{L^4_{t,x}}\,ds\\
\lesssim &\sum_{j\le k} b_jb_k2^{j+k}\int_0^\infty (1+s2^{2k})^{-3}(s2^{2k})^{-3/8}\,ds\\
&+\sum_{j\le k} b_kb_j2^{(k+j)/2}2^j\int_0^\infty (1+s2^{2k})^{-4}(s2^{2j})^{-3/8}\,ds\\
&+\sum_{j\ge k} b_j^22^{k-j}2^{2j}\int_0^\infty (1+s2^{2j})^{-7}(s2^{2j})^{-3/8}\,ds.
\end{aligned}$$

Call the integrals I_1, I_2, and I_3, respectively. Clearly I_1 and I_3 satisfy $I_1 \lesssim 2^{-2k}$ and $I_3 \lesssim 2^{-2j}$. By Cauchy-Schwarz, I_2 satisfies

$$\begin{aligned}
I_2 &\lesssim \left(\int_0^\infty (1+s2^{2k})^{-8}(1+s2^{2j})^4\,ds\right)^{1/2}\left(\int_0^\infty (1+s2^{2j})^{-4}(s2^{2j})^{-3/8}\,ds\right)^{1/2}\\
&\lesssim 2^{-j-k}.
\end{aligned}$$

Therefore

$$\begin{aligned}\|P_k A_m(0)\|_{L^4_{t,x}} &\lesssim b_k \sum_{j\le k}\left(b_j 2^{j-k} + b_j 2^{(j-k)/2}\right) + \sum_{j\ge k} b_j^2 2^{k-j}\\ &\lesssim b_k^2.\end{aligned}$$

□

4.2. **The electric potential is perturbative.**

Proposition 4.7. *The term $V_m = (A_t + A_x^2)\psi_m - i\psi_\ell \mathrm{Im}(\bar{\psi}_\ell \psi_m)$ is perturbative.*

We start with the cubic term. Here we use off-diagonal decay and take advantage of the fact that any ψ may be placed in a G_k space. Hence Lemma 3.14 applies and from Corollary 3.15 we conclude

$$\|P_k(\psi_\ell \mathrm{Im}(\bar{\psi}_\ell \psi_m))\|_{N_k(T)} \lesssim \varepsilon b_k.$$

We move on to $A_x^2\psi$.

We conclude as a corollary of Lemma 4.6 that

$$\|A_x^2(0)\|_{L^2_{t,x}} \lesssim \|A_x(0)\|^2_{L^4_{t,x}} \lesssim \sum_{k\in\mathbf{Z}} \|P_k A_x(0)\|^2_{L^4_{t,x}} \lesssim \sup_{j\in\mathbf{Z}} b_j^2 \cdot \sum_{k\in\mathbf{Z}} b_k^2. \tag{4.12}$$

We next show how to use this L^2 bound to control $A_x^2\psi_m$ in N_k spaces.

Lemma 4.8. *Let $f \in L^2_{t,x}$. Then*

$$\|P_k(f\psi_m)\|_{N_k(T)} \lesssim \|f\|_{L^2_{t,x}} b_k. \tag{4.13}$$

Proof. Begin with the following Littlewood-Paley decomposition of $P_k(f\psi_x)$:

$$\begin{aligned}P_k(f\psi_x) &= P_k(P_{<k-80} f P_{k-5<\cdot<k+5}\psi_x)\\ &\quad + \sum_{\substack{|k_1-k|\le 4\\ k_2\le k-80}} P_k(P_{k_1} f P_{k_2}\psi_x) + \sum_{\substack{|k_1-k_2|\le 90\\ k_1,k_2>k-80}} P_k(P_{k_1} f P_{k_2}\psi_x).\end{aligned}$$

The first term is controlled using (3.3):

$$\begin{aligned}\|P_k(P_{<k-80} f P_{k-5<\cdot<k+5}\psi_x)\|_{N_k(T)} &\le \|P_k(P_{<k-80} f P_{k-5<\cdot<k+5}\psi_x)\|_{L^{4/3}_{t,x}}\\ &\le \|P_{<k-80} f\|_{L^2_{t,x}} \|P_{k-5<\cdot<k+5}\psi_x\|_{G_k(T)}.\end{aligned}$$

To control the second term we apply (3.4):

$$\|P_k(P_{k_1} f P_{k_2}\psi_x)\|_{N_k(T)} \lesssim 2^{-|k_2-k|/6} \|P_{k_1} f\|_{L^2_{t,x}} \|P_{k_2}\psi_x\|_{G_{k_2}(T)}.$$

To control the high-high interaction, apply (3.5):

$$\|P_k(P_{k_1} f P_{k_2}\psi_x)\|_{N_k(T)} \lesssim 2^{-|k-k_2|/6} \|P_{k_1} f\|_{L^2_{t,x}} \|P_{k_2}\psi_x\|_{G_{k_2}(T)}.$$

Therefore

$$\sum_{\substack{|k_1-k_2|\leq 90\\ k_1,k_2>k-80}} \|P_k(P_{k_1} f P_{k_2}\psi_x)\|_{N_k(T)} \lesssim \sum_{\substack{|k_1-k_2|\leq 90\\ k_1,k_2>k-80}} 2^{-|k-k_2|/6}\|P_{k_1} f\|_{L^2_{t,x}} b_{k_2},$$

and so by Cauchy-Schwarz

$$\sum_{\substack{|k_1-k_2|\leq 90\\ k_1,k_2>k-80}} \|P_k(P_{k_1} f P_{k_2}\psi_x)\|_{N_k(T)} \lesssim b_k \left(\sum_{k_1\geq k-80} \|P_{k_1} f\|^2_{L^2_{t,x}}\right)^{1/2},$$

Upon interchanging the $L^2_{t,x}$ and ℓ^2 norms, we conclude from the standard square function estimate that

$$\sum_{\substack{|k_1-k_2|\leq 90\\ k_1,k_2>k-80}} \|P_k(P_{k_1} f P_{k_2}\psi_x)\|_{N_k(T)} \lesssim \|f\|_{L^2_{t,x}} b_k.$$

□

Together (4.12) and Lemma 4.8 imply

$$\|P_k(A_x^2\psi_m)\|_{N_k(T)} \lesssim \varepsilon b_k. \tag{4.14}$$

We proceed to analyze $A_t\psi$, the leading term of the electric potential. This term requires more effort to bound, behaving as a blend of the cubic term and $A_x^2\psi$. The main difficulty arises from the fact that we do not control $\psi_t(s)$ in F_k spaces for positive heat flow times $s > 0$. While at $s = 0$ we do indeed have $\psi_t = iD_j\psi_j$ (because at $s = 0$ it holds that $\psi_t = i\psi_s$) as a consequence of the fact that ϕ is a Schrödinger map, along the heat flow we do not have such an explicit representation of ψ_t and instead must access it through the commutator of the heat and Schrödinger flows. Thus our first step is to represent ψ_t as $i\psi_s + \Psi$. It may be tempting to try to place Ψ in the F_k spaces, as we do have such bounds for ψ_s. The F_k bounds for ψ_s, however, are obtained indirectly and as a consequence of bounds on ψ_x and A_x. A different approach would therefore be required to bound Ψ in F_k. The bounds on $\Psi(s), s > 0$ that can be readily obtained are those in L^4. Owing to the fact that $\Psi(0) = 0$, it turns out that under the assumption of energy dispersion, these bounds come with an ε gain. Representing A_t as

$$A_t(s) = -\int_0^\infty \mathrm{Im}(i\bar{\psi}_s\psi_s)(s')ds' - \int_0^\infty \mathrm{Im}(\bar{\Psi}\psi_s)(s')ds', \tag{4.15}$$

we show that the L^2 norm of the second integral is small, enabling us to treat its contribution to $A_t\psi$ using Lemma 4.8.

Now $\psi_s = D_j\psi_j$ and as already mentioned does enjoy bounds in the F_k-spaces. The effect of the integration is cancel the derivatives that appear, and so in principle the first integral in (4.15) is on par with the cubic term $\psi_\ell \mathrm{Im}(\bar{\psi}_\ell\psi_m)$. However, any one of the terms in the cubic term may be placed

in a G_k space, whereas, for $s' > 0$, we cannot place $\psi_s(s')$ in G_k spaces. For this reason we further decompose ψ_s, representing it as $\psi_s = \partial_\ell \psi + iA_\ell \psi_\ell$. Rewriting the first integral in (4.15) as

$$-\int_0^\infty |\psi_s|^2(s')ds',$$

we see that it suffices (by Young's inequality) to just consider the contributions from $|\partial_\ell \psi|^2$ and from $|A_\ell \psi_\ell|^2$. For the latter term, we pull out A_ℓ in L^2 (which is largest at $s = 0$), and control $\int_0^\infty |\psi_\ell|^2 \lesssim_{E_0} 1$ using heat flow bounds; hence this term has a contribution like that of $A_x^2 \psi$. For the $|\partial_\ell \psi|^2$ term, we expand $\psi(s)$ using the Duhamel formula as the sum of its linear evolution and nonlinear evolution. While we do not propagate bounds in the G_k-spaces along the nonlinear heat flow, such bounds do in fact propagate along the linear flow; the contribution from the three linear terms taken together therefore is on par with that of the cubic term. The nonlinear evolution terms must be dealt with more delicately, as here we must resort to placing these in F_k-spaces, resulting in worse off-diagonal gains. The upshot, however, is that these terms come with an energy-dispersion gain that is enough to offset the consequences of inferior decay.

We turn to the details.

Lemma 4.9. *It holds that*

$$\|\int_0^\infty (\overline{\Psi} \cdot D_\ell \psi_\ell)(s)\; ds\|_{L^2_{t,x}} \lesssim \varepsilon(1 + \sum_j b_j^2)^2 \sum_k b_k^2.$$

Proof. We first bound $(\overline{\Psi} \cdot D_\ell \psi_\ell)(s)$ in L^2. Define

$$\mu_k(s) := \sup_{k' \in \mathbf{Z}} 2^{-\delta|k-k'|} \|P_k \Psi(s)\|_{L^4_{t,x}} \quad \text{and} \quad \nu_k(s) := \sup_{k' \in \mathbf{Z}} 2^{-\delta|k-k'|} \|P_k (D_\ell \psi_\ell)(s)\|_{L^4_{t,x}},$$

then

$$\|(\overline{\Psi} \cdot D_\ell \psi_\ell)(s)\|_{L^2_{t,x}} \lesssim \sum_k \mu_k(s) \sum_{j \leq k} \nu_j(s) + \sum_k \nu_k(s) \sum_{j \leq k} \mu_j(s). \tag{4.16}$$

From Corollaries 4.5 and 4.4 it follows that

$$\mu_k(s) \lesssim \varepsilon(1 + s2^{2k})^{-2} 2^k (1 + \sum_p b_p^2) b_k \tag{4.17}$$

and

$$\nu_k(s) \lesssim (1 + s2^{2k})^{-2} 2^k (1 + \sum_p b_p^2) b_k. \tag{4.18}$$

Therefore

$$\int_0^\infty \|(\overline{\Psi}\cdot D_\ell\psi_\ell)(s)\|_{L^2_{t,x}}\, ds \lesssim \int_0^\infty \sum_k \mu_k(s) \sum_{j\le k} \nu_j(s) ds$$
$$\lesssim \varepsilon(1+\sum_p b_p^2)^2 \sum_k 2^k b_k \sum_{j\le k} 2^j b_j \times$$
$$\times \int_0^\infty (1+s2^{2j})^{-2}(1+s2^{2k})^{-2}\, ds$$

Then

$$\sum_{j\le k} 2^j b_j \int_0^\infty (1+s2^{2j})^{-2}(1+s2^{2k})^{-2}\, ds \lesssim \int_0^\infty (1+s2^{2k})^{-2}\, ds \sum_{j\le k} 2^j b_j$$

and

$$\int_0^\infty (1+s2^{2k})^{-2}\, ds \lesssim 2^{-2k}$$

so that

$$\sum_k 2^k b_k \sum_{j\le k} 2^j b_j \int_0^\infty (1+s2^{2j})^{-2}(1+s2^{2k})^{-2}\, ds \lesssim \sum_k b_k^2.$$

□

Lemma 4.10. *It holds that*

$$\|\int_0^\infty |A_\ell\psi_\ell|^2(s)ds\|_{L^2_{t,x}} \lesssim \sup_j b_j^2 \sum_k b_k^2.$$

Proof. Start by taking a Littlewood-Paley decomposition of $|A_\ell|^2$:

$$\int_0^\infty |A_\ell\psi_\ell|^2(s)ds = \int_0^\infty \sum_{j,k} P_j A_\ell(s)\cdot P_k\overline{A}_\ell(s)\cdot|\psi_x|^2(s)ds$$

We bound this expression in absolute value by

$$\sum_{j,k} \sup_{s\ge 0}|P_j A_x(s)|\cdot \sup_{s'\ge 0}|P_k A_x(s')|\cdot \int_0^\infty |\psi_x|^2(r)dr.$$

Next we take the $L^2_{t,x}$ norm, which we control by placing the integral in $L^\infty_{t,x}$ and the summation in $L^2_{t,x}$. We control the summation by

$$\sum_{j,k} \|\sup_{s\ge 0}|P_j A_x(s)|\cdot \sup_{s'\ge 0}|P_k A_x(s')|\|_{L^2_{t,x}} \lesssim \sum_j \|\sup_{s\ge 0}|P_j A_x(s)|\|^2_{L^4_{t,x}}$$

Now

$$\sup_{s\ge 0}|P_j A_m(s)| \lesssim \sup_{s\ge 0}\int_s^\infty |P_j(\overline{\psi_m(s')}D_\ell\psi_\ell(s'))|ds' \le \int_0^\infty |P_j(\overline{\psi_m(s')}D_\ell\psi_\ell(s'))|ds',$$

and in view of the proof of Lemma 4.6, the L^2 norm of the right hand side is bounded by b_j^2.

Thus it remains to show

$$\| \int_0^\infty |\psi_\ell|^2(s) ds \|_{L^\infty_{t,x}} \lesssim 1.$$

For fixed s, x, t, however, ψ_ℓ is simply the representation of $\partial_\ell \phi$ with respect to a certain orthonormal basis of $T_{\phi(s,x,t)}\mathbf{S}^2$. Therefore $|\psi_\ell| = |\partial_\ell \phi|$ and so we may invoke (the uniform in time) estimate (2.14). $\square$

Lemma 4.11. *It holds that*

$$P_k \left(\int_0^\infty |\partial_\ell \psi_\ell|^2(s') ds' \psi_m(0) \right) \lesssim \varepsilon b_k \tag{4.19}$$

Proof. We write

$$\psi_\ell(s) = e^{s\Delta}\psi_\ell(0) + \int_0^s e^{(s-r)\Delta} U_\ell(r) dr$$

and expand (4.19) accordingly. Additionally, we perform a Littlewood-Paley decomposition of each term. The trilinear term

$$P_k \left(\sum_{k_1,k_2,k_3} \int_0^\infty e^{s\Delta} P_{k_1} \partial_\ell \psi_\ell(0) \cdot e^{s\Delta} P_{k_2} \partial_\ell \psi_\ell(0) \; ds \cdot P_{k_3} \psi_m(0) \right)$$

we bound by in $N_k(T)$ by

$$\sum_{\substack{(k_1,k_2,k_3)\in \\ Z_1(k)\cup Z_2(k)\cup Z_3(k)}} C_{k,k_1,k_2,k_3} 2^{k_1} \|P_{k_1}\psi_x(0)\|_{G_{k_1}(T)} 2^{k_2} \|P_{k_2}\psi_x(0)\|_{G_{k_2}(T)} \times$$
$$\times \|P_{k_3}\psi_m(0)\|_{G_{k_3}(T)} \int_0^\infty (1 + s2^{2k_1})^{-20}(1 + s2^{2k_2})^{-20} ds,$$

where C_{k,k_1,k_2,k_3} is as in Lemma 3.14 and Z_1, Z_2, Z_3 are given by (3.12). As

$$2^{k_1} 2^{k_2} \int_0^\infty (1 + s2^{2k_1})^{-20}(1 + s2^{2k_2})^{-20} ds \lesssim 1,$$

it suffices to control

$$\sum_{\substack{(k_1,k_2,k_3)\in \\ Z_1(k)\cup Z_2(k)\cup Z_3(k)}} C_{k,k_1,k_2,k_3} \|P_{k_1}\psi_x(0)\|_{G_{k_1}(T)} \times \tag{4.20}$$
$$\times \|P_{k_2}\psi_x(0)\|_{G_{k_2}(T)} \|P_{k_3}\psi_m(0)\|_{G_{k_3}(T)}.$$

We invoke Corollary 3.15 to bound (4.20) by b_k^3, which suffices for this term in view of the energy dispersion assumption.

Next we must consider products involving either one or two $\int_0^s e^{(s-r)\Delta} U_\ell(r) dr$ terms. The arguments are similar to that of the case already considered. Here, however, we only place $P_{k_3}\psi_m(0)$ in a G_k space; the remaining two terms we control in F_k spaces using either (4.2) or (4.5). Hence in our $N_k(T)$ bound we use C_{k,k_1,k_2,k_3} as in Lemma 3.10 rather than Lemma 3.14, and to

sum we must use Corollary 3.11 instead of Corollary 3.15. Corollary 3.11, however, only supplies the bound over $\mathbf{Z}^3 \setminus Z_2(k)$.

On Z_2 the sum is controlled by

$$\begin{aligned}
\varepsilon \sum_{(k_1,k_2,k_3)\in Z_2(k)} 2^{-|k-k_3|/6} b_{k_1} b_{k_2} b_{k_3} &\lesssim \varepsilon \sum_{k,k_3 \le k_1-40} 2^{-|k-k_3|/6} b_{k_1} b_{k_1} b_{k_3} \\
&\lesssim \varepsilon b_k \sum_{k,k_3\le k_1-40} 2^{-|k-k_3|/6} 2^{\delta|k-k_3|} b_{k_1}^2 \\
&\lesssim \varepsilon b_k \sum_{k_1 \ge k+40} b_{k_1}^2 .
\end{aligned}$$

□

4.3. **Decomposing the magnetic potential.** We begin by introducing a paradifferential decomposition of the magnetic nonlinearity, splitting it into two pieces. This decomposition depends upon a frequency parameter $k \in \mathbf{Z}$, which we suppress in the notation; this same k will also be the output frequency whose behavior we are interested in controlling. It also depends upon the frequency gap parameter $\varpi \in \mathbf{Z}_+$. How ϖ is chosen and the exact role it plays are discussed in §5.2. There it is shown that ϖ may be set equal to a sufficiently large universal constant.

Define $A_{\mathrm{lo}\wedge\mathrm{lo}}$ as

$$A_{m,\mathrm{lo}\wedge\mathrm{lo}}(s) := -\sum_{k_1,k_2\le k-\varpi} \int_s^\infty \operatorname{Im}(\overline{P_{k_1}\psi_m} P_{k_2}\psi_s)(s')ds'$$

and $A_{\mathrm{hi}\vee\mathrm{hi}}$ as

$$A_{m,\mathrm{hi}\vee\mathrm{hi}}(s) := -\sum_{\max\{k_1,k_2\}>k-\varpi} \int_s^\infty \operatorname{Im}(\overline{P_{k_1}\psi_m} P_{k_2}\psi_s)(s')ds'$$

so that $A_m = A_{m,\mathrm{lo}\wedge\mathrm{lo}} + A_{m,\mathrm{hi}\vee\mathrm{hi}}$. Similarly define $B_{\mathrm{lo}\wedge\mathrm{lo}}$ as

$$B_{m,\mathrm{lo}\wedge\mathrm{lo}} := -i\sum_{k_3} (\partial_\ell(A_{\ell,\mathrm{lo}\wedge\mathrm{lo}} P_{k_3}\psi_m) + A_{\ell,\mathrm{lo}\wedge\mathrm{lo}}\partial_\ell P_{k_3}\psi_m)$$

and $B_{\mathrm{hi}\vee\mathrm{hi}}$ as

$$B_{m,\mathrm{hi}\vee\mathrm{hi}} := -i\sum_{k_3} (\partial_\ell(A_{\ell,\mathrm{hi}\vee\mathrm{hi}} P_{k_3}\psi_m) + A_{\ell,\mathrm{hi}\vee\mathrm{hi}}\partial_\ell P_{k_3}\psi_m)$$

so that $B_m = B_{m,\mathrm{lo}\wedge\mathrm{lo}} + B_{m,\mathrm{hi}\vee\mathrm{hi}}$.

Our goal is to control $P_k B_m$ in $N_k(T)$. We consider first $P_k B_{m,\mathrm{hi}\vee\mathrm{hi}}$, performing a trilinear Littlewood-Paley decomposition. In order for frequencies k_1, k_2, k_3 to have an output in this expression at a frequency k, we must have $(k_1,k_2,k_3) \in Z_2(k) \cup Z_3(k) \cup Z_0(k)$, where $Z_0(k) := Z_1(k) \cap \{(k_1,k_2,k_3) \in$

$\mathbf{Z}^3 : k_1, k_2 > k - \varpi\}$ and Z_2, Z_3 are given by (3.12). We apply Lemma 3.10 to bound $P_k B_{m,\mathrm{hi}\vee\mathrm{hi}}$ in $N_k(T)$ by

$$\sum_{\substack{(k_1,k_2,k_3)\in \\ Z_2(k)\cup Z_3(k)\cup Z_0(k)}} \int_0^\infty 2^{\max\{k,k_3\}} C_{k,k_1,k_2,k_3} \|P_{k_1}\psi_x(s)\|_{F_{k_1}} \times \\ \times \|P_{k_2}(D_\ell \psi_\ell(s))\|_{F_{k_2}} \|P_{k_3}\psi_m(0)\|_{G_{k_3}} ds,$$

which, thanks to (4.2) and (4.3), is controlled by

$$\sum_{\substack{(k_1,k_2,k_3)\in \\ Z_2(k)\cup Z_3(k)\cup Z_0(k)}} 2^{\max\{k,k_3\}} C_{k,k_1,k_2,k_3} b_{k_1} b_{k_2} b_{k_3} \times \\ \times \int_0^\infty (1+s2^{2k_1})^{-4} 2^{k_2} (s2^{2k_2})^{-3/8} (1+s2^{2k_2})^{-2} ds.$$

As

$$\int_0^\infty (1+s2^{2k_1})^{-4} 2^{k_2} (s2^{2k_2})^{-3/8} (1+s2^{2k_2})^{-2} ds \lesssim 2^{-\max\{k_1,k_2\}}, \tag{4.21}$$

we reduce to

$$\sum_{\substack{(k_1,k_2,k_3)\in \\ Z_2(k)\cup Z_3(k)\cup Z_0(k)}} 2^{\max\{k,k_3\}-\max\{k_1,k_2\}} C_{k,k_1,k_2,k_3} b_{k_1} b_{k_2} b_{k_3}.$$

To estimate $P_k B_{m,\mathrm{hi}\vee\mathrm{hi}}$ on $Z_2 \cup Z_3$, we apply Corollary 3.12 and use the energy dispersion hypothesis. As for $Z_0(k)$, we note that its cardinality $|Z_0(k)|$ satisfies $|Z_0(k)| \lesssim \varpi$ independently of k. Hence for fixed ϖ summing over this set is harmless given sufficient energy dispersion.

Consider now the leading term $P_k B_{m,\mathrm{lo}\wedge\mathrm{lo}}$. Bounding this in N_k with any hope of summing requires the full strength of the decay that comes from the local smoothing/maximal function estimates. Such bounds as are immediately at our disposal (i.e., (3.10) and (3.11), however, do not bring $B_{m,\mathrm{lo}\wedge\mathrm{lo}}$ within the perturbative framework, instead yielding a bound of the form

$$\sum_{\substack{k_1,k_2 \le k-\varpi \\ |k_3-k|\le 4}} b_{k_1} b_{k_2} b_{k_3},$$

which is problematic since even $\sum_{j\ll k} c_j^2 \sim E_0^2 = O(1)$ for k large enough. This stands in sharp contrast with the small energy setting.

In the next section, however, we are able to capture enough improvement in such estimates so as to barely bring $B_{m,\mathrm{lo}\wedge\mathrm{lo}}$ back within reach of our bootstrap approach.

5. Local smoothing and bilinear Strichartz

In this section we establish local smoothing and bilinear Strichartz estimates for solutions to certain magnetic nonlinear Schrödinger equations. These are in the spirit of [40, 39, 59]. We shall then apply these to the paralinearized derivative field equations written with respect to the caloric gauge.

We introduce some notation. Let $I_k(\mathbf{R}^d)$ denotes the set $\{\xi \in \mathbf{R}^d : |\xi| \in [-2^{k-1}, 2^{k+1}]\}$ and $I_{(-\infty,k]} := \bigcup_{j \leq k} I_j$. For a d-vector-valued function $B = (B_\ell)$ on $\mathbf{R}^d$ with real entries, define the magnetic Laplacian Δ_B, acting on complex-valued functions f, via

$$\Delta_B f := (\partial_x + iB)((\partial_x + iB)f) = \Delta f + i(\partial_\ell B_\ell)f + 2iB_\ell \partial_\ell f - B_\ell^2 f. \tag{5.1}$$

For a unit vector $\mathbf{e} \in \mathbf{S}^{d-1}$, denote by $\{x \cdot \mathbf{e} = 0\}$ the orthogonal complement in $\mathbf{R}^d$ of the span of $\mathbf{e}$, equipped with the induced measure. Given $\mathbf{e}$, we can construct a positively oriented orthonormal basis $\mathbf{e}, \mathbf{e}_1, \ldots, \mathbf{e}_{d-1}$ of $\mathbf{R}^d$ so that $\mathbf{e}_1, \ldots, \mathbf{e}_{d-1}$ form an orthonormal basis for $\{x \cdot \mathbf{e} = 0\}$. For complex-valued functions f on $\mathbf{R}^d$, define $E_{\mathbf{e}}(f) : \mathbf{R} \to \mathbf{R}$ as

$$E_{\mathbf{e}}(f)(x_0) := \int_{x \cdot \mathbf{e} = 0} |f|^2 dx' = \int_{\mathbf{R}^{d-1}} |f(x_0 \mathbf{e} + x_j \mathbf{e}_j)|^2 dx', \tag{5.2}$$

where the implicit sum runs over $1, 2, \ldots, d-1$, and dx' is standard $d-1$-dimensional Lebesgue measure. We also adopt the following notation for this section: for z, ζ complex,

$$z \wedge \zeta := z\bar{\zeta} - \bar{z}\zeta = 2i\mathrm{Im}(z\bar{\zeta}).$$

5.1. The key lemmas.

Lemma 5.1 (Abstract almost-conservation of energy). *Let $d \geq 1$ and $\mathbf{e} \in \mathbf{S}^{d-1}$. Let v be a $C_t^\infty(H_x^\infty)$ function on $\mathbf{R}^d \times [0, T)$ solving*

$$(i\partial_t + \Delta_{\mathcal{A}})v = \Lambda_v \tag{5.3}$$

with initial data v_0. Take $\mathcal{A}_\ell$ to be real-valued, smooth, and bounded, with $\Delta_{\mathcal{A}}$ defined via (5.1). Then

$$\|v\|_{L_t^\infty L_x^2}^2 \leq \|v_0\|_{L_x^2}^2 + \left| \int_0^T \int_{\mathbf{R}^d} v \wedge \Lambda_v dx dt \right|. \tag{5.4}$$

Proof. We begin with

$$\frac{1}{2}\partial_t \int |v|^2 dx = \int \mathrm{Re}(\bar{v}\partial_t v) dx,$$

which may equivalently be written as

$$i\partial_t \int |v|^2 dx = -\int v \wedge i\partial_t v dx.$$

Substituting from (5.3) yields

$$i\partial_t \int |v|^2 dx = \int v \wedge (\Delta_{\mathcal{A}} v - \Lambda_v)\, dx.$$

Expanding $\Delta_{\mathcal{A}}$ using (5.1) and using the straightforward relations

$$\partial_\ell (v \wedge i\mathcal{A}_\ell v) = v \wedge i(\partial_\ell \mathcal{A}_\ell) v + v \wedge 2i\mathcal{A}_\ell \partial_\ell v$$

and

$$\partial_\ell (v \wedge \partial_\ell v) = v \wedge \Delta v,$$

we get

$$\begin{aligned} i\partial_t \int |v|^2 dx = & \int \partial_\ell (v \wedge \partial_\ell v) dx + \int \partial_\ell (v \wedge i\mathcal{A}_\ell v) dx \\ & - \int v \wedge \mathcal{A}_\ell^2 v dx - \int v \wedge \Lambda_v dx. \end{aligned}$$

The first two terms on the right hand side vanish upon integration in x; the third is equal to zero because $\mathcal{A}_\ell^2$ is real. Integrating in time and taking absolute values therefore yields

$$\left| \int_{\mathbf{R}^d} |v(T')|^2 - |v_0|^2 dx \right| = \left| \int_0^{T'} \int_{\mathbf{R}^d} v \wedge \Lambda_v dx dt \right|$$

for any time $T' \in (0, T]$. □

Lemma 5.2 (Local smoothing preparation)**.** *Let $d \geq 1$ and $\mathbf{e} \in \mathbf{S}^{d-1}$. Let $j, k \in \mathbf{Z}$ and $j = k + O(1)$. Let $\varepsilon_m > 0$ be a small positive number such that $\varepsilon_m 2^{O(1)} \ll 1$. Let v be a $C_t^\infty(H_x^\infty)$ function on $\mathbf{R}^d \times [0, T)$ solving*

$$(i\partial_t + \Delta_{\mathcal{A}}) v = \Lambda_v, \tag{5.5}$$

where $\mathcal{A}_\ell$ is real-valued, smooth, and satisfies the estimate

$$\|\mathcal{A}\|_{L^\infty_{t,x}} \leq \varepsilon_m 2^k. \tag{5.6}$$

The solution v is assumed to have (spatial) frequency support in I_k, with the additional constraint that $\mathbf{e} \cdot \xi \in [2^{j-1}, 2^{j+1}]$ for all ξ in the support of $\hat{v}$. Then

$$2^j \int_0^T E_{\mathbf{e}}(v) dt \lesssim \|v\|^2_{L_t^\infty L_x^2} + \left| \int_0^T \int_{x \cdot \mathbf{e} \geq 0} v \wedge \Lambda_v dx dt \right| + 2^j \int_0^T E_{\mathbf{e}}(v + i 2^{-j} \partial_{\mathbf{e}} v) dt. \tag{5.7}$$

Proof. We begin by introducing

$$M_{\mathbf{e}}(t) := \int_{x \cdot \mathbf{e} \geq 0} |v(x,t)|^2 dx.$$

Then

$$0 \leq M_{\mathbf{e}}(t) \leq \|v(t)\|^2_{L_x^2(\mathbf{R}^d)} \leq \|v\|^2_{L_t^\infty L_x^2([-T,T] \times \mathbf{R}^d)}. \tag{5.8}$$

Differentiating in time yields

$$\begin{aligned} i\dot{M}_{\mathbf{e}}(t) &= \int_{x\cdot\mathbf{e}\geq 0} v \wedge (i\partial_t v)dx \\ &= \int_{x\cdot\mathbf{e}\geq 0} v \wedge (\Delta_{\mathcal{A}} v - \Lambda_v)dx, \end{aligned}$$

which may be rewritten as

$$i\dot{M}_{\mathbf{e}}(t) = \int_{x\cdot\mathbf{e}\geq 0} \partial_\ell (v \wedge (\partial_\ell + i\mathcal{A}_\ell)v)dx - \int_{x\cdot\mathbf{e}\geq 0} v \wedge \Lambda_v dx. \tag{5.9}$$

By integrating by parts,

$$\int_{x\cdot\mathbf{e}\geq 0} \partial_\ell (v \wedge (\partial_\ell + i\mathcal{A}_\ell)v)dx = -\int_{x\cdot\mathbf{e}=0} v \wedge (\partial_{\mathbf{e}} v + i\mathbf{e}\cdot\mathcal{A}v)dx',$$

and therefore (5.9) may be rewritten as

$$-\int_{x\cdot\mathbf{e}=0} v \wedge (\partial_{\mathbf{e}} v + i\mathbf{e}\cdot\mathcal{A}v)dx' = i\dot{M}_{\mathbf{e}}(t) + \int_{x\cdot\mathbf{e}\geq 0} v \wedge \Lambda_v dx. \tag{5.10}$$

On the one hand, we have the heuristic that $\partial_{\mathbf{e}} v \approx i2^j v$ since v has localized frequency support. On the other hand, since $\mathcal{A}$ is real-valued, we have

$$\int_0^T \int_{x\cdot\mathbf{e}=0} v \wedge i\mathbf{e}\cdot\mathcal{A}v dx' dt = 2\int_0^T \int_{x\cdot\mathbf{e}=0} \mathbf{e}\cdot\mathcal{A}|v|^2 dx' dt \tag{5.11}$$

and hence by assumption (5.6) also

$$\int_0^T \int_{x\cdot\mathbf{e}=0} |\mathcal{A}||v|^2 dx' dt \leq \varepsilon_m 2^k \int_0^T \int_{x\cdot\mathbf{e}=0} |v|^2 dx' dt. \tag{5.12}$$

Together these facts motivate rewriting $v \wedge \partial_e v$ as

$$v \wedge \partial_{\mathbf{e}} v = 2\cdot i2^j |v|^2 + v \wedge (\partial_{\mathbf{e}} v - i2^j v). \tag{5.13}$$

Using (5.11), (5.13), and the bounds (5.12) and (5.8) in (5.10), we obtain by time-integration that

$$\begin{aligned} (1-\varepsilon_m 2^{k-j})2^j \int_0^T E_{\mathbf{e}}(v)dt \leq \|v\|^2_{L^\infty_t L^2_x} + \left| \int_0^T \int_{x\cdot\mathbf{e}\geq 0} v \wedge \Lambda_v dx dt \right| \\ + 2\cdot 2^j \int_0^T \int_{x\cdot\mathbf{e}=0} |v + i2^{-j}\partial_{\mathbf{e}} v||v| dx' dt. \end{aligned}$$

Applying Cauchy-Schwarz to the last term yields

$$2^j \int_0^T \int_{x\cdot\mathbf{e}=0} |v+i2^{-j}\partial_{\mathbf{e}} v||v| dx' dt \leq 8\cdot 2^j \int_0^T E_{\mathbf{e}}(v+i2^{-j}\partial_{\mathbf{e}} v)dt + \frac{1}{8}\cdot 2^j \int_0^T E_{\mathbf{e}}(v)dt.$$

Therefore (5.7). □

We now describe the constraints on the nonlinearity that we shall require in the abstract setting

Definition 5.3. Let $\mathcal{P}$ be a fixed compact subset of $\{1 < p < \infty\}$. A a bilinear form $B(\cdot,\cdot)$ is said to be *adapted* to $\mathcal{P}$ provided it measures its arguments in Strichartz-type spaces, the estimate

$$\left|\int_0^T \int_{\mathbf{R}^d} f \wedge g dx dt\right| \lesssim B(f,g)$$

holds for all complex-valued functions f, g on $\mathbf{R}^d \times [0,T]$, Bernstein's inequalities hold in both arguments of B, and these arguments are measured in L^p_x only for $p \in \mathcal{P}$. Given $B(\cdot,\cdot)$, and $\mathbf{e} \in \mathbf{S}^{d-1}$, we define $B_{\mathbf{e}}(\cdot,\cdot)$ via

$$B_{\mathbf{e}}(f,g) := B(f, \chi_{\{x\cdot\mathbf{e}\geq 0\}} g).$$

Moreover, we assume that B is such that

$$B_{\mathbf{e}}(f,g) \leq B(f,g)$$

for all f, g.

Definition 5.4. Let $\mathbf{e} \in \mathbf{S}^{d-1}$ and let $\mathcal{A}_\ell$ be real-valued and smooth. Let v be a $C^\infty_t(H^\infty_x)$ function on $\mathbf{R}^d \times [0,T)$ solving

$$(i\partial_t + \Delta_{\mathcal{A}})v = \Lambda_v.$$

Assume v is (spatially) frequency-localized to I_k with the additional constraint that $\mathbf{e}\cdot\xi \in [2^{j-1}, 2^{j+1}]$ for all ξ in the support of $\hat{v}$. Define a sequence of functions $\{v^{(m)}\}_{m=1}^\infty$ by setting $v^{(1)} = v$ and

$$v^{(m+1)} := v^{(m)} + i2^{-j}\partial_{\mathbf{e}} v^{(m)}.$$

By (5.1) and the Leibniz rule,

$$(i\partial_t + \Delta_{\mathcal{A}})v^{(m)} = \Lambda_{v^{(m)}},$$

where

$$\Lambda_{v^{(m)}} := (1+i2^{-j}\partial_{\mathbf{e}})\Lambda_{v^{(m-1)}} + i2^{-j}(i\partial_{\mathbf{e}}\partial_\ell A_\ell - \partial_{\mathbf{e}} A_\ell^2)v^{(m-1)} - 2^{-j+1}(\partial_{\mathbf{e}} A_\ell)\partial_\ell v^{(m-1)}.$$

The sequence $\{v^{(m)}\}_{m=1}^\infty$ is called the *derived sequence* corresponding to v.

Suppose we are given an adapted form B. The derived sequence is said to be *controlled* with respect to B provided that $B(v^{(m)}, \Lambda_{v^{(m)}}) < \infty$ for each $m \geq 1$.

We remark that if the derived sequence $\{v^{(m)}\}_{m=1}^\infty$ of v is controlled, then for all $\ell \geq 1$, the derived sequences $\{v^{(m)}\}_{m=\ell}^\infty$ are also controlled.

Theorem 5.5 (Abstract local smoothing)**.** *Let $d \geq 1$ and $\mathbf{e} \in \mathbf{S}^{d-1}$. Let $j, k \in \mathbf{Z}$ and $j = k + O(1)$. Let $\varepsilon_m > 0$ be a small positive number such that $\varepsilon_m 2^{O(1)} \ll 1$. Let $\eta > 0$. Let $\mathcal{P}$ be a fixed compact subset of $(1,\infty)$ with $2 \in \mathcal{P}$, and let $B_{\mathbf{e}}$ be a form adapted to $\mathcal{P}$. Let v be a $C^\infty_t(H^\infty_x)$ function on $\mathbf{R}^d \times [0,T)$ solving*

$$(i\partial_t + \Delta_{\mathcal{A}})v = \Lambda_v, \tag{5.14}$$

where $\mathcal{A}_\ell$ is real-valued, smooth, has spatial Fourier support in $I_{(-\infty,k]}$, and satisfies the estimate

$$\|\mathcal{A}\|_{L^\infty_{t,x}} \leq \varepsilon_m 2^k. \tag{5.15}$$

The solution v is assumed to have (spatial) frequency support in I_k. We take Λ_v to be frequency-localized to $I_{(-\infty,k]}$.

Assume moreover that

$$\mathbf{e} \cdot \xi \in [(1-\eta)2^j, (1+\eta)2^j] \tag{5.16}$$

for all ξ in the support of $\hat{v}$.

If the derived sequence of v is controlled with respect to B, then there exist $\eta^* > 0$ such that, for all $0 \leq \eta < \eta^*$, the local smoothing estimate

$$2^j \int_0^T E_{\mathbf{e}}(v)dt \lesssim \|v\|^2_{L^\infty_t L^2_x} + B(v, \Lambda_v) \tag{5.17}$$

holds uniformly in T and $j = k + O(1)$.

Proof. The foundation for proving (5.17) is (5.7), which for an adapted form B implies

$$2^j \int_0^T E_{\mathbf{e}}(v)dt \lesssim \|v\|^2_{L^\infty_t L^2_x} + B(v, \Lambda_v) + 2^j \int_0^T E_{\mathbf{e}}(v + i2^{-j}\partial_{\mathbf{e}} v)dt. \tag{5.18}$$

Since $B_{\mathbf{e}}(v, \Lambda_v) \leq B(v, \Lambda_v)$, our goal is control the last term in (5.18). This we do using a bootstrap argument that hinges upon the fact that $\tilde{v} := v + i2^{-j}\partial_{\mathbf{e}} v$ is the second term in the derived sequence of v, and that being "controlled" is an inherited property (in the sense of the comments following Definition 5.4).

By Bernstein's and Hölder's inequalities, we have

$$2^j \int_0^T E_{\mathbf{e}}(v)dt \lesssim 2^{2j} T \|v\|^2_{L^\infty_t L^2_x}.$$

for any v. For fixed $T > 0$ and $k \in \mathbf{Z}$, let $K_{T,k} \geq 1$ be the best constant for which the inequality

$$2^j \int_0^T E_{\mathbf{e}}(v)dt \leq K_{T,k} \left(\|v\|^2_{L^2_x} + B(v, \Lambda_v) \right) \tag{5.19}$$

holds for all controlled sequences. Applying (5.19) to $\tilde{v}$ results in

$$2^j \int_0^T E_{\mathbf{e}}(\tilde{v})dt \leq K_{T,k} \left(\|\tilde{v}\|^2_{L^2_x} + B(\tilde{v}, \Lambda_{\tilde{v}}) \right), \tag{5.20}$$

and thus we seek to control norms of $\tilde{v}$ in terms of those of v.

Let $\widetilde{P}_k, \widetilde{P}_{j,\mathbf{e}}$ denote slight fattenings of the Fourier multipliers $P_k, P_{j,\mathbf{e}}$. On the one hand, Plancherel implies

$$\|(1 + i2^{-j}\partial_{\mathbf{e}})\widetilde{P}_{j,\mathbf{e}}\widetilde{P}_k\|_{L^2_x \to L^2_x} \lesssim \eta. \tag{5.21}$$

On the other hand, Bernstein's inequalities imply

$$\|(1+i2^{-j}\partial_{\mathbf{e}})\widetilde{P}_{j,\mathbf{e}}\widetilde{P}_k\|_{L^p_x\to L^p_x}\lesssim 1, \qquad 1\le p\le\infty.$$

Therefore it follows from Riesz-Thorin interpolation that

$$\|(1+i2^{-j}\partial_{\mathbf{e}})\widetilde{P}_{j,\mathbf{e}}\widetilde{P}_k\|_{L^p_x\to L^p_x}\lesssim \begin{cases} \eta^{2/p} & 2\le p<\infty \\ \eta^{2-2/p} & 1<p\le 2. \end{cases}$$

Restricting to $p\in\mathcal{P}$, we conclude that there exists a $q>0$ such that

$$\|(1+i2^{-j}\partial_{\mathbf{e}})\widetilde{P}_{j,\mathbf{e}}\widetilde{P}_k\|_{L^p_x\to L^p_x}\lesssim \eta^q \tag{5.22}$$

for all $p\in\mathcal{P}$ and all η small enough.

Applying (5.22) and Bernstein to $\tilde{v}$ yields

$$\|\tilde{v}\|_{L^2_x}\lesssim \eta^q\|v\|_{L^2_x}, \qquad B(\tilde{v},\Lambda_{\tilde{v}})\lesssim \eta^q B(v,\Lambda_v),$$

which, combined with (5.20) and (5.18), leads to

$$2^j\int_0^T E_{\mathbf{e}}(v)dt\lesssim (1+\eta^q K_{T,k})\left(\|v\|^2_{L^\infty_t L^2_x}+B(v,\Lambda_v)\right).$$

As $K_{T,k}$ is the best constant for which (5.19) holds, it follows that

$$K_{T,k}\lesssim 1+\eta^q K_{T,k}$$

and hence that $K_{T,k}\lesssim 1$ for η small enough. □

Corollary 5.6. *Given the assumptions of Theorem 5.5, it holds that*

$$2^j\int_0^T E_{\mathbf{e}}(v)dt\lesssim \|v_0\|^2_{L^2_x}+B(v,\Lambda_v)$$

Proof. This is an immediate consequence of Theorem 5.5 and Lemma 5.1. □

Corollary 5.7 (Abstract bilinear Strichartz)**.** *Let $d\ge 1$ and $\mathbf{e}\in\mathbf{S}^{d-1}$. Set $\tilde{\mathbf{e}}=(-\mathbf{e},\mathbf{e})/\sqrt{2}$. Let $j,k\in\mathbf{Z}$ and $j=k+O(1)$. Let $\varepsilon_m>0$ be a small positive number such that $\varepsilon_m 2^{O(1)}\ll 1$. Let $\eta>0$. Let $\mathcal{P}$ be a fixed compact subset of $(1,\infty)$ with $2\in\mathcal{P}$, and let $B_{\tilde{\mathbf{e}}}$ be a form that is adapted to $\mathcal{P}$.*

Let $w(x,y)$ be a $C^\infty_t(H^\infty_{x,y})$ function on $\mathbf{R}^{2d}\times[0,T)$,equal to w_0 at $t=0$ and solving

$$(i\partial_t+\Delta_{\mathcal{A}})w=\Lambda_w,$$

where $\mathcal{A}_{k'}$ is real-valued, smooth, has spatial Fourier support in $I_{(-\infty,k]}$, and satisfies the estimate

$$\|\mathcal{A}\|_{L^\infty_{t,x,y}}\le\varepsilon_m 2^k.$$

Assume w has (spatial) frequency support in I_k and that

$$\tilde{\mathbf{e}}\cdot\xi\in[(1-\eta)2^j,(1+\eta)2^j]$$

for all ξ in the support of $\hat{w}$. Take Λ_w to be frequency-localized to $I_{(-\infty,k]}$.

Suppose that $w(x,y)$ admits a decomposition $w(x,y) = u(x)v(y)$, where u has frequency support in I_ℓ, $\ell \ll k$. Use u_0, v_0 to denote $u(t=0), v(t=0)$. If the derived sequence of w is controlled with respect to B, then

$$\|uv\|_{L^2_{t,x}}^2 \lesssim 2^{\ell(d-1)} 2^{-j} \left(\|u_0\|_{L^2_x}^2 \|v_0\|_{L^2_x}^2 + B(w, \Lambda_w) \right) \tag{5.23}$$

uniformly in T and $j = k + O(1)$ provided η is small enough.

Proof. Taking into account that

$$\|w_0\|_{L^2_{x,y}} = \|u_0\|_{L^2_x} \|v_0\|_{L^2_x},$$

we apply Corollary 5.6 to w at $(x,y) = 0$:

$$2^j \int_0^T E_{\tilde{\mathbf{e}}}(w) dt \lesssim \|u_0\|_{L^2_x}^2 \|v_0\|_{L^2_x}^2 + B(w, \Lambda_w). \tag{5.24}$$

We complete $(-\mathbf{e}, \mathbf{e})/\sqrt{2}$ to a basis as follows:

$$(-\mathbf{e}, \mathbf{e})/\sqrt{2}, (0, \mathbf{e}_1), \ldots, (0, \mathbf{e}_{d-1}), (\mathbf{e}, \mathbf{e})/\sqrt{2}, (\mathbf{e}_1, 0), \ldots, (\mathbf{e}_{d-1}, 0).$$

On the one hand, $E_{\tilde{\mathbf{e}}}(w)(0)$ is by definition (see (5.2)) equal to

$$\int_{\mathbf{R}} \int_{\mathbf{R}^{2d-2}} |u(0 \cdot \mathbf{e} + r\mathbf{e} + x_j \mathbf{e}_j, t) v(0 \cdot \mathbf{e} + r\mathbf{e} + y_j \mathbf{e}_j, t)|^2 dx' dy' dr.$$

We rewrite it as

$$\int_{\mathbf{R}} \int_{\mathbf{R}^{d-1}} |v(r\mathbf{e} + y_j \mathbf{e}_j, t)|^2 dy' \int_{\mathbf{R}^{d-1}} |u(r\mathbf{e} + x_j \mathbf{e}_j, t)|^2 dx' dr. \tag{5.25}$$

On the other hand,

$$\begin{aligned} \|uv\|_{L^2_y}^2 &= \int_{\mathbf{R}^d} |u(y,t)|^2 |v(y,t)|^2 dy \\ &= \int_{\mathbf{R}} \int_{\mathbf{R}^{d-1}} |u(r\mathbf{e} + y_j \mathbf{e}_j)|^2 |v(r\mathbf{e} + y_j \mathbf{e}_j)|^2 dy' dr, \end{aligned}$$

and by applying Bernstein to u in the y' variables, we obtain

$$\|uv\|_{L^2_y}^2 \lesssim 2^{\ell(d-1)} \int_{\mathbf{R}} \int_{\mathbf{R}^{d-1}} |v(r\mathbf{e} + y_j \mathbf{e}_j)|^2 dy' \int_{\mathbf{R}^{d-1}} |u(r\mathbf{e} + x_j \mathbf{e}_j)|^2 dx' dr. \tag{5.26}$$

Together (5.26), (5.25), and (5.24) imply (5.23). □

5.2. **Applying the abstract lemmas.** We would like to apply the abstract estimates just developed to the evolution equation (2.7). We work in the caloric gauge and adopt the magnetic potential decomposition introduced in §4.3. Throughout $\varepsilon > 0$ is assumed to be a very small number such that $b_k \leq \varepsilon$ and $\varepsilon^{1/2} \sum_j b_j^2 \ll 1$; in the next section we set it equal to a suitable power of the energy dispersion parameter ε_0.

Our starting point is the equation

$$(i\partial_t + \Delta)\psi_m = B_{m,\mathrm{lo}\wedge\mathrm{lo}} + B_{m,\mathrm{hi}\vee\mathrm{hi}} + V_m. \tag{5.27}$$

Applying Fourier multipliers P_k, $P_{j,\theta}P_k$, or variants thereof, we easily obtain corresponding evolution equations for $P_k\psi_m$, $P_{j,\theta}P_k$, etc. In rewriting a projection P of (5.27) in the form (5.3), evidently $\Delta_{\mathcal{A}}\psi_m$ should somehow come from $\Delta P\psi_m - PB_{m,\mathrm{lo}\wedge\mathrm{lo}}$, whereas $PB_{m,\mathrm{hi}\vee\mathrm{hi}} + PV_m$ ought to constitute the leading part of the nonlinearity Λ. Fourier multipliers P, however, do not commute with the connection coefficients A, and therefore in order to use the abstract machinery we must first track and control certain commutators. Toward this end we adopt some notation from [51].

Following [51, §1], we use $L_O(f_1,\dots,f_m)(s,x,t)$ to denote any multi-linear expression of the form

$$L_O(f_1,\dots,f_m)(s,x,t)$$
$$:= \int K(y_1,\dots,y_{M(c)}f_1(s,x-y_1,t)\dots f_m(s,x-y_{M(c)},t)dy_1\dots dy_{M(c)},$$

where the kernel K is a measure with bounded mass (and K may change from line to line). Moreover, the kernel of L_O does not depend upon the index α. Also, we extend this notation to vector or matrices by making K into an appropriate tensor. The expression $L_O(f_1,\dots,f_m)$ may be thought of as a variant of $O(f_1,\dots,f_m)$. It obeys two key properties. The first is simple consequence of Minkowski's inequality (e.g., see [51, Lemma 1]).

Lemma 5.8. *Let $X_1,\dots,X_m,X$ be spatially translation-invariant Banach spaces such that the product estimate*

$$\|f_1\cdots f_m\|_X \le C_0\|f_1\|_{X_1}\cdots\|f_m\|_{X_m}$$

holds for all scalar-valued $f_i \in X_i$ and for some constant $C_0 > 0$. Then

$$\|L_O(f_1,\dots,f_m)\|_X \lesssim (Cd)^{Cm}C_0\|f_1\|_{X_1}\cdots\|f_m\|_{X_m}$$

holds for all $f_i \in X_i$ that are scalars, d-dimensional vectors, or $d\times d$ matrices.

The next lemma is an adapation of [51, Lemma 2].

Lemma 5.9 (Leibniz rule)**.** *Let P_k' be a C^∞ Fourier multiplier whose frequency support lies in some compact subset of $I_k(\mathbf{R}^d)$. The commutator identity*

$$P_k'(fg) = fP_k'g + L_O(\partial_x f, 2^{-k}g)$$

holds.

Proof. Rescale so that $k=0$ and let $m(\xi)$ denote the symbol of P_0' so that

$$\widehat{P_0'h}(\xi) := m(\xi)\hat{h}(\xi).$$

By the Fundamental Theorem of Calculus, we have

$$\begin{aligned}\left(P_0'(fg) - fP_0'g\right)(s,x,t) &= \int_{\mathbf{R}^d} \check{m}(y)\left(f(s,x-y,t) - f(s,x,t)\right) g(s,x-y,t)dy \\ &= -\int_0^1 \int_{\mathbf{R}^d} \check{m}(y) y \cdot \partial_x f(s,x-ry,t) g(s,x-y,t) dy dr.\end{aligned}$$

The conclusion follows from the rapid decay of $\hat{m}$. $\square$

We are interested in controlling $P_{j,\theta} P_k \psi_m$ in $L_\theta^{\infty,2}$ over all $\theta \in \mathbf{S}^1$ and $|j - k| \leq 20$. In the abstract framework, however, we assumed a much tighter localization than $P_{j,\theta}$ provides. Therefore we decompose $P_{j,\theta}$ as a sum

$$P = \sum_{l=1,\ldots,O((\eta^*)^{-1}} P_{j,\theta,l}, \tag{5.28}$$

and it suffices by the triangle inequality to bound $P_{j,\theta,l} P_k \psi_m$.

For notational convenience set $P := P_{j,\theta,l} P_k$. Applying P to (5.27) yields

$$(i\partial_t + \Delta) P\psi_m = P\left(B_{m,\mathrm{lo}\wedge\mathrm{lo}} + B_{m,\mathrm{hi}\vee\mathrm{hi}} + V_m\right).$$

Now

$$PB_{m,\mathrm{lo}\wedge\mathrm{lo}} = -iP \sum_{|k_3 - k| \leq 4} (\partial_\ell (A_{\ell,\mathrm{lo}\wedge\mathrm{lo}} P_{k_3} \psi_m) + A_{\ell,\mathrm{lo}\wedge\mathrm{lo}} \partial_\ell P_{k_3} \psi_m),$$

as P localizes to a region of the annulus I_k. Applying Lemma 5.9, we obtain

$$PB_{m,\mathrm{lo}\wedge\mathrm{lo}} = -i(\partial_\ell (A_{\ell,\mathrm{lo}\wedge\mathrm{lo}} P\psi_m) - iA_{\ell,\mathrm{lo}\wedge\mathrm{lo}} \partial_\ell P\psi_m) + R$$

where

$$R := \sum_{|k_3 - k| \leq 4} \Big(L_O(\partial_x \partial_\ell A_{\ell,\mathrm{lo}\wedge\mathrm{lo}}, 2^{-k} P_{k_3} \psi_m) + L_O(\partial_x A_{\ell,\mathrm{lo}\wedge\mathrm{lo}}, 2^{-k} P_{k_3} \partial_\ell \psi_m) \Big). \tag{5.29}$$

Set

$$\mathcal{A}_m := A_{m,\mathrm{lo}\wedge\mathrm{lo}}.$$

Then

$$(i\partial_t + \Delta_{\mathcal{A}}) P\psi_m = P(B_{m,\mathrm{hi}\vee\mathrm{hi}} + V_m) + \mathcal{A}_x^2 P\psi_m + R. \tag{5.30}$$

It is this equation that we shall show fits within the abstract local smoothing framework.

First we check that Lemmas 5.1 and 5.2 apply. The main condition to check is (5.6). Key are (2.15) and Bernstein, which together with the fact that $\mathcal{A}$ is frequency-localized to $I_{(-\infty,k]}$ provide the estimate

$$\|\mathcal{A}\|_{L^\infty_{t,x}} \lesssim 2^k.$$

To achieve the ε_m gain, we adjust ϖ, which forces a gap between I_k and the frequency support of $\mathcal{A}$, i.e., we localize $\mathcal{A}$ to $I_{(-\infty,k-\varpi]}$ instead. Thus it suffices to set $\varpi \in \mathbf{Z}_+$ equal to a sufficiently large universal constant.

There is more to check in showing that (5.30) falls within the purview of Theorem 5.5. Already we have $d = 2$, $\mathbf{e} = \theta$, $\varepsilon_m \sim 2^{-\varpi}$, $\mathcal{A}_m := A_{m,\mathrm{lo}\wedge\mathrm{lo}}$, $v = P_{j,\theta,l}P_k\psi_m$, and $\Lambda_v = P(B_{m,\mathrm{hi}\vee\mathrm{hi}} + V_m) + \mathcal{A}_x^2 P\psi_m + R$.

Next we choose $\mathcal{P}$ based upon the norms used in N_k, with the exception of the local smoothing/maximal function estimates. To be precise, define the new norms $\widetilde{N}_k$ via

$$\|f\|_{\widetilde{N}_k(T)} := \inf_{f=f_1+f_2+f_3+f_4+f_5} \|f_1\|_{L^{4/3}_{t,x}} + 2^{k/6}\|f_2\|_{L^{3/2,6/5}_{\hat{\theta}_1}} + 2^{k/6}\|f_3\|_{L^{3/2,6/5}_{\hat{\theta}_2}} + 2^{-k/6}\|f_4\|_{L^{6/5,3/2}_{\hat{\theta}_1}} + 2^{-k/6}\|f_5\|_{L^{6/5,3/2}_{\hat{\theta}_2}}$$

and similarly $\widetilde{G}_k$ via

$$\|f\|_{\widetilde{G}_k(T)} := \|f\|_{L^\infty_t L^2_x} + \|f\|_{L^4_{t,x}} + 2^{-k/2}\|f\|_{L^4_x L^\infty_t} + 2^{-k/6}\sup_{\theta\in\mathbf{S}^1}\|f\|_{L^{3,6}_\theta} + 2^{k/6}\sup_{|j-k|\leq 20}\sup_{\theta\in\mathbf{S}^1}\|P_{j,\theta}f\|_{L^{6,3}_\theta}.$$

Set $\mathcal{P} = \{2, 3, 3/2, 4, 4/3, 6, 5/6\}$. We define the form $B(\cdot,\cdot)$ via

$$B(f, g) := \|f\|_{\widetilde{G}_k(T)}\|g\|_{\widetilde{N}_k(T)} \tag{5.31}$$

and B_θ by

$$B_\theta(f, g) := B(f, \chi_{\{x\cdot\theta\geq 0\}}g) \tag{5.32}$$

as in Definition 5.3. That B is adapted to $\mathcal{P}$ is a direct consequence of the definition.

Proposition 5.10. *Let $\eta > 0$ be a parameter to be specified later. Let $d = 2$, $\mathbf{e} = \theta$, $\varepsilon_m \sim 2^{-\varpi}$, $\mathcal{A}_m := A_{m,\mathrm{lo}\wedge\mathrm{lo}}$, $v = P^{(\eta)}_{j,\theta,l}P_k\psi_m$, $\Lambda_v = P(B_{m,\mathrm{hi}\vee\mathrm{hi}}+V_m)+\mathcal{A}_x^2 P\psi_m + R$, and $\mathcal{P} = \{2, 3, 3/2, 4, 4/3, 6, 5/6\}$. Let B, B_θ be given by (5.31) and (5.32) respectively. Then the conditions of Theorem 5.5 are satisfied and the derived sequence of v is controlled with respect to B so that conclusion (5.17) holds for $v = P^{(\eta)}_{j,\theta,l}P_k\psi_m$ given η sufficiently small.*

Proof. The only claim of Proposition 5.10 that remains to be verified is that the derived sequence of $v = P_{j,\theta,l}P_k\psi_m$ is controlled with respect to B. In particular, we need to show that for each $q \geq 1$ we have

$$B(v^{(q)}, \Lambda_{v^{(q)}}) < \infty,$$

where $v^{(1)} := P_{j,\theta,l}P_k\psi_m$,

$$v^{(q+1)} := v^{(q)} + i2^{-j}\partial_\theta v^{(q)},$$

and

$$\Lambda_{v^{(q+1)}} := (1 + i2^{-j}\partial_\theta)\Lambda_{v^{(q)}} + i2^{-j}(i\partial_\theta\partial_\ell\mathcal{A}_\ell - \partial_\theta\mathcal{A}_\ell^2)v^{(q)} - 2^{-j+1}(\partial_\theta\mathcal{A}_\ell)\partial_\ell v^{(q)}.$$

We first prove the following lemma.

Lemma 5.11. *The right hand side of (5.30) satisfies*

$$\|P(B_{m,\mathrm{hi}\vee\mathrm{hi}} + V_m) + \mathcal{A}_x^2 P\psi_m + R\|_{\widetilde{N}_k(T)} \lesssim c_k + \varepsilon b_k$$

Proof. We will repeatedly use implicitly the fact that the multiplier $P_{j,\theta,l}$ is bounded on $L^p, 1 \le p \le \infty$, so that in particular P obeys estimates that are at least as good as those obeyed by P_k.

From §4 it follows that $P_k(B_{m,\mathrm{hi}\vee\mathrm{hi}} + V_m)$ is perturbative and bounded in $\widetilde{N}_k(T)$ by $c_k + \varepsilon b_k$. The $\widetilde{N}_k(T)$ estimates on PV_m immediately imply the boundedness of $\mathcal{A}_x^2 P\psi_m$.

To estimate R, we apply Lemma 3.10 to bound $PB_{m,\mathrm{lo}\wedge\mathrm{lo}}$ by

$$\sum_{(k_1,k_2,k_3)\in Z_1(k)} \int_0^\infty 2^{\max\{k_1,k_2\}} 2^{k_3-k} C_{k,k_1,k_2,k_3} \|P_{k_1}\psi_x(s)\|_{F_{k_1}} \times \\ \times \|P_{k_2}(D_\ell\psi_\ell(s))\|_{F_{k_2}} \|P_{k_3}\psi_m(0)\|_{G_{k_3}} ds,$$

which, in view of (4.2), (4.3), and (4.21), is controlled by

$$\sum_{(k_1,k_2,k_3)\in Z_1(k)} C_{k,k_1,k_2,k_3} b_{k_1} b_{k_2} b_{k_3}.$$

Summation is achieved thanks to Corollary 3.11. □

We return to the proof of the proposition, and in particular to showing that $B(v,\Lambda_v) < \infty$. We apply Lemma 5.11 to control Λ_v in $\widetilde{N}_k$. Since by assumption $P\psi_m$ is bounded in $\widetilde{G}_k(T)$ (even in $G_k(T)$), we conclude that $B(v,\Lambda_v) < \infty$.

Next we need to show $B(v^q, \Lambda_{v^q}) < \infty$ for $q > 1$. By Bernstein,

$$\|v^{(q)}\|_{\widetilde{G}_k(T)} \lesssim \|v^{(q-1)}\|_{\widetilde{G}_k(T)}.$$

Similarly,

$$\|(1 + i2^{-j})\partial_\theta \Lambda_{v^{(q)}}\|_{\widetilde{N}_k(T)} \lesssim \|\Lambda_{v^{(q-1)}}\|_{\widetilde{N}_k(T)}.$$

Thus it remains to control $i2^{-j}(i\partial_\theta\partial_\ell\mathcal{A}_\ell - \partial_\theta\mathcal{A}_\ell^2)v^{(q)}$ and $2^{-j+1}(\partial_\theta\mathcal{A}_\ell)\partial_\ell v^{(q)}$ in $\widetilde{N}_k$ for each $q > 1$. Both are consequences of arguments in Lemma 5.11: Boundedness of $2^{-j}(\partial_\theta\partial_\ell\mathcal{A}_\ell)v^{(q)}$ and $2^{-j+1}(\partial_\theta\mathcal{A}_\ell)\partial_\ell v^{(q)}$ follows directly from the argument used to control R and from Bernstein's inequality, whereas boundedness of $2^{-j}(\partial_\theta\mathcal{A}_\ell^2)v^{(q)}$ is a consequence of Bernstein and the estimates on $\mathcal{A}_x^2 P\psi_m$. □

Combining Lemma 5.11 and Proposition 5.10, we conclude that Corollary 5.6 applies to $v = P\psi_m$, with right hand side bounded by $c_k^2 + \varepsilon b_k^2$. In view of the decomposition (5.28), we conclude

Corollary 5.12. *The function $P_k\psi_m$ satisfies*

$$\sup_{|j-k|\le 20}\sup_{\theta\in\mathbf{S}^1}\|P_{j,\theta}P_k\psi_m\|_{L^{\infty,2}_\theta}\lesssim 2^{-k/2}(c_k+\varepsilon^{1/2}b_k).$$

Our next objective is to apply Corollary 5.7 to the case where w splits as a product $u(x)v(y)$ where u,v are appropriate frequency localizations of ψ_m or $\overline{\psi_m}$. First we must find function spaces suitable for defining an adapted form. We start with $(i\partial_t+\Delta_{\mathcal{A}})w=\Lambda_w$ and observe how it behaves with respect to separation of variables. If $w(x,y)=u(x)v(y)$, then the left hand side may be rewritten as $u\cdot(i\partial_t+\Delta_{\mathcal{A}_y})v+v\cdot(i\partial_t+\Delta_{\mathcal{A}_x})u$. Let $\Lambda_u:=(i\partial_t+\Delta_{\mathcal{A}_x})u$ and $\Lambda_v:=(i\partial_t+\Delta_{\mathcal{A}_y})v$. Then

$$(i\partial_t+\Delta_{\mathcal{A}})(uv)=u\Lambda_v+v\Lambda_u.$$

We control

$$\int_0^T\int_{\mathbf{R}^2\times\mathbf{R}^2}u(x)v(y)\left(\Lambda_u(x)v(y)+u(x)\Lambda_v(y)\right)dxdydt$$

as follows: in the case of the first term $u(x)v(y)\Lambda_u(x)v(y)$ we place each $v(y)$ in $L^\infty_tL^2_y$; we bound $u(x)\Lambda_u(x)$ by placing $u(x)$ in G_j and $\Lambda_u(x)$ in $\widetilde{N}_j$. To control $u(x)v(y)u(x)\Lambda_v(y)$, we simply reverse the roles of u and v (and of x and y). This leads us to the spaces $\overline{N}_{k,\ell}$ defined by

$$\|f\|_{\overline{N}_{k,\ell}(T)}:=\inf_{\substack{J\in\mathbf{Z}_+,f(x,y)=\\ \sum_{j=1}^{2J}(g_j(x)h_j(y)+g_{j+1}(x)h_{j+1}(y))}}\Big(\|g_j\|_{\widetilde{N}_\ell(T)}\|h_j\|_{L^\infty_tL^2_y}+\|g_{j+1}\|_{L^\infty_tL^2_x}\|h_{j+1}\|_{\widetilde{N}_k(T)}\Big),\tag{5.33}$$

and the spaces $\overline{G}_{k,\ell}$ defined via

$$\|f\|_{\overline{G}_{k,\ell}(T)}:=\|\|f(x,y)\|_{\widetilde{G}_k(T)(y)}\|_{\widetilde{G}_\ell(T)(x)}.\tag{5.34}$$

We use these spaces to define the form $\overline{B}(\cdot,\cdot)$ by

$$\overline{B}(f,g):=\|f\|_{\overline{G}_{k,\ell}(T)}\|g\|_{\overline{N}_{k,\ell}(T)}\tag{5.35}$$

and the form $\overline{B}_\Theta$ by

$$\overline{B}_\Theta(f,g):=\overline{B}(f,\chi_{\{(x,y)\cdot\Theta\ge 0\}}g),\tag{5.36}$$

where $\Theta:=(-\theta,\theta)$.

Proposition 5.13. *Let $\eta>0$ be a small parameter and $\varpi\in\mathbf{Z}_+$ a large parameter, both to be specified later. Let $j,k,\ell\in\mathbf{Z}$, $j=k+O(1)$, $\ell\ll k$. Let $d=2$, $\mathbf{e}=\theta$, $\varepsilon_m\sim 2^{-\varpi}$, $\mathcal{A}_x:=A_{m,\mathrm{lo}\wedge\mathrm{lo}}$, $v=P^{(\eta)}_{j,\theta,l}P_k\psi_m$, $\Lambda_v=P(B_{m,\mathrm{hi}\vee\mathrm{hi}}+V_m)+\mathcal{A}_x^2P\psi_m+R$, and $\mathcal{P}=\{2,3,3/2,4,4/3,6,5/6\}$. Here R is given by (5.29). Also, let $u=\overline{P_\ell\psi_p}$, $p\in\{1,2\}$ and $\Lambda_u=P_\ell(B_{p,\mathrm{hi}\vee\mathrm{hi}}+$*

$V_p) + \mathcal{A}_x^2 P_\ell \psi_p + R'$, where R' is given by (5.29), but defined in terms of derivative field ψ_ℓ and frequency ℓ rather than ψ_m and k.

Let $w(x,y) := u(x)v(y)$, $\mathcal{A} := (\mathcal{A}_x, \mathcal{A}_y)$, $\Lambda_w := \Lambda_u v + u\Lambda_v$. Then, for ϖ sufficiently large and η sufficiently small, the conditions of Corollary 5.7 are satisfied and (5.23) applies to $u(x)v(x)$.

Proof. The frequency support conditions on $\mathcal{A}$ and Λ_w are easily verified. That the L^∞ bound on $\mathcal{A}$ holds follows from (2.15) and Bernstein provided ϖ is large enough (cf. discussion preceding Proposition 5.10). In order to guarantee the frequency support conditions on w, it is necessary to make the gap $\ell \ll k$ sufficiently large with respect to η.

That $\overline{B}$ is adapted to $\mathcal{P}$ is a straightforward consequence of its definition. To see that the derived sequence of w is controllable, we look to the proof of Proposition 5.10 and the definitions of the $\overline{N}_{k,\ell}$, $\overline{G}_{k,\ell}$ spaces. □

In a spirit similar to that of the proof of Corollary 5.12, we may combine Lemma 5.11 and the proof of Proposition 5.10 to control $B(w, \Lambda_w)$; in fact, in measuring Λ_w in the $\overline{N}_{k,\ell}$ spaces, it suffices to take $J = 1$ (see (5.33)). Then we obtain $B(w, \Lambda_w) \lesssim \varepsilon b_j b_k$. Using decomposition (5.28) and the triangle inequality to bound $P_k \psi_m$ in terms of the bounds on $P_{j,\theta,\ell}^{(\eta)} P_k \psi_m$, we obtain the bilinear Strichartz analogue of Corollary 5.12. In our application, however, the lower-frequency term will not simply be $\overline{P_j \psi_\ell}$, but rather its heat flow evolution $\overline{P_j \psi_\ell}(s)$.

Corollary 5.14 (Improved Bilinear Strichartz)**.** *Let $j, k \in \mathbf{Z}$, $j \ll k$, and $u \in \{P_j \psi_\ell, \overline{P_j \psi_\ell} : j \leq k - \varpi, \ell \in \{1,2\}\}$. Then for $s \geq 0$,*

$$\|u(s) P_k \psi_m(0)\|_{L^2_{t,x}} \lesssim 2^{(j-k)/2} (1 + s2^{2j})^{-8} \left(c_j c_k + \varepsilon b_j b_k\right). \tag{5.37}$$

Proof. It only remains to prove (5.37) when $s > 0$. Let $v := P_k \psi_m$. Using the Duhamel formula, we write

$$u(s)v = (e^{s\Delta} u(0)) v(0) + \int_0^s e^{(s-s')\Delta} U(s')\, ds' \cdot v(0), \tag{5.38}$$

where U is defined by (2.22) in terms of u.

To control the nonlinear term $\int_0^s e^{(s-s')\Delta} U(s')\, ds' \cdot v(0)$ in L^2 we apply local smoothing estimate (3.11), which places the nonlinear evolution in $F_j(T)$ and $v(0)$ in $G_k(T)$. Using (4.5) to bound the $F_j(T)$ norm, we conclude

$$\| \int_0^s e^{(s-s')\Delta} \widetilde{U}(s')\, ds' \cdot v(0) \|_{L^2_{t,x}} \lesssim \varepsilon 2^{(j-k)/2} (1 + s2^{2j})^{-4} b_j b_k. \tag{5.39}$$

It remains to show

$$\|(e^{s\Delta} u) v\|_{L^2_{t,x}} \lesssim (1 + s2^{2j})^{-4} 2^{(j-k)/2} (c_j c_k + \varepsilon b_j b_k), \tag{5.40}$$

which is not a direct consequence of the time $s = 0$ bound. Let $\mathcal{T}_a$ denote the spatial translation operator that acts on functions $f(x,t)$ according to $\mathcal{T}_a f(x,t) := f(x-a,t)$. Then if

$$\|(\mathcal{T}_{x_1} u)(\mathcal{T}_{x_2} v)\|_{L^2_{t,x}} \lesssim 2^{(j-k)/2}(c_j c_k + \varepsilon b_j b_k) \tag{5.41}$$

can be shown to hold for all $x_1, x_2 \in \mathbf{R}^2$ then (5.40) follows from Minkowski's and Young's inequalities.

Consider, then, a solution w to

$$(i\partial_t + \Delta_{\mathcal{A}}(x,t))w(x,t) = \Lambda_w(x,t)$$

satisfying the conditions of Theorem 5.5. The translate $\mathcal{T}_{x_0} w(x,t)$ then satisfies

$$(i\partial_t + \Delta_{\mathcal{T}_{x_0}(\mathcal{A})(x,t)})(\mathcal{T}_{x_0} w)(x,t) = (\mathcal{T}_{x_0}\Lambda_w)(x,t).$$

The operator $\mathcal{T}_{x_0}$ clearly does not affect $L^\infty_{t,x}$ bounds or frequency support conditions. Moreover, since the norms used to define B are translation-invariant, the adapted form B controls the derived sequence of $\mathcal{T}_{x_0} w$. Therefore Proposition 5.13 holds for spatial translates of frequency projections of ψ_m, from which we conclude (5.41). □

6. Concluding the proofs of the main theorems

6.1. **Closing the gauge field bootstrap.** We turn first to the completion of the proof of Theorem 1.5, as we now have in place all of the estimates that we need to prove (1.15).

Using the main linear estimate of Proposition 3.6 and the decomposition introduced in §4.3, we obtain

$$\begin{aligned} \|P_k \psi_m\|_{G_k(T)} \lesssim \|P_k \psi_m(0)\|_{L^2_x} &+ \|P_k V_m\|_{N_k(T)} \\ &+ \|P_k B_{m,\mathrm{hi}\vee\mathrm{hi}}\|_{N_k(T)} + \|P_k B_{m,\mathrm{lo}\wedge\mathrm{lo}}\|_{N_k(T)}. \end{aligned} \tag{6.1}$$

In §4 it was shown that $P_k V_m$ and $P_k B_{m,\mathrm{hi}\vee\mathrm{hi}}$ are perturbative in the sense that

$$\|P_k V_m\|_{N_k(T)} + \|P_k B_{m,\mathrm{hi}\vee\mathrm{hi}}\|_{N_k(T)} \lesssim \varepsilon b_k,$$

where $\varepsilon > 0$ is assumed to satisfy $b_k \leq \varepsilon$ and $\varepsilon^{1/2}\sum_j b_j^2 \ll 1$ In view of (1.12) and the bootstrap hypothesis (1.13), we set $\varepsilon := \varepsilon_0^{9/10}$. Clearly this satisfies $b_k \leq \varepsilon$. Moreover,

$$\varepsilon^{1/2}\sum_j b_j^2 \leq \varepsilon_0^{3/10}\sum_j c_j^2 \lesssim_{E_0} \varepsilon_0^{3/10}.$$

To handle $P_k B_{m,\mathrm{lo}\wedge\mathrm{lo}}$, we first write

$$P_k B_{m,\mathrm{lo}\wedge\mathrm{lo}} = -i\partial_\ell(A_{\ell,\mathrm{lo}\wedge\mathrm{lo}} P_k \psi_m) + R,$$

where R is a perturbative remainder (thanks to a slight modification of Lemma 5.11). Therefore

$$\|P_k\psi_m\|_{G_k(T)} \lesssim c_k + \varepsilon b_k + \|\partial_\ell(A_{\ell,\mathrm{lo}\wedge\mathrm{lo}}P_k\psi_m)\|_{N_k(T)}. \tag{6.2}$$

Thus it remains to control $-i\partial_\ell(A_{\ell,\mathrm{lo}\wedge\mathrm{lo}}P_k\psi_m)$, which we expand as

$$-iP_k\partial_\ell \sum_{\substack{k_1,k_2\leq k-\varpi \\ |k_3-k|\leq 4}} \int_0^\infty \mathrm{Im}(\overline{P_{k_1}\psi_\ell}P_{k_2}\psi_s)(s')P_{k_3}\psi_m(0)ds', \tag{6.3}$$

and whose $N_k(T)$ norm we denote by N_{lo}. The key now is to apply Corollary 5.14 to $\overline{P_{k_1}\psi_\ell}(s')$ and $P_{k_3}\psi_m(0)$, after first placing all of (6.3) in $N_k(T)$ using (3.10). We obtain

$$\begin{aligned} N_{\mathrm{lo}} &\lesssim 2^k \sum_{\substack{k_1,k_2\leq k-\varpi \\ |k_3-k|\leq 4}} 2^{-|k-k_2|/2}2^{-|k_1-k_3|/2}2^{-\max\{k_1,k_2\}}b_{k_2}\left(c_{k_1}c_{k_3}+\varepsilon b_{k_1}b_{k_3}\right) \\ &\lesssim 2^k \sum_{k_1,k_2\leq k-\varpi} 2^{(k_1+k_2)/2-k}2^{-\max\{k_1,k_2\}}b_{k_2}(c_{k_1}c_k+\varepsilon b_{k_1}b_k) \end{aligned}$$

Without loss of generality we restrict the sum to $k_1 \leq k_2$:

$$\sum_{k_1\leq k_2\leq k-\varpi} 2^{(k_1-k_2)/2}b_{k_2}(c_{k_1}c_k+\varepsilon b_{k_1}b_k)$$

Using the frequency envelope property to sum off the diagonal, we reduce to

$$N_{\mathrm{lo}} \lesssim \sum_{j\leq k-\varpi}(b_jc_jc_k+\varepsilon b_j^2 b_k).$$

Combining this with (6.2) and the fact that R is perturbative, we obtain

$$b_k \lesssim c_k + \varepsilon b_k + \sum_{j\leq k-\varpi}(b_jc_jc_k+\varepsilon b_j^2 b_k),$$

which, in view of our choice of ε, reduces to

$$b_k \lesssim c_k + c_k\sum_{j\leq k-\varpi} b_jc_j.$$

Squaring and applying Cauchy-Schwarz yields

$$b_k^2 \lesssim (1+\sum_{j\leq k-\varpi} b_j^2)c_k^2. \tag{6.4}$$

Setting

$$B_k := 1+\sum_{j<k} b_j^2$$

in (6.4) leads to

$$B_{k+1} \leq B_k(1+Cc_k^2)$$

with $C > 0$ independent of k. Therefore

$$B_{k+m} \le B_k \prod_{\ell=1}^{m} (1 + C c_{k+\ell}^2) \le B_k \exp(C \sum_{\ell=1}^{m} c_{k+\ell}^2) \lesssim_{E_0} B_k.$$

Since $B_k \to 1$ as $k \to -\infty$, we conclude

$$B_k \lesssim_{E_0} 1$$

uniformly in k, so that, in particular,

$$\sum_{j \in \mathbf{Z}} b_j^2 \lesssim 1, \tag{6.5}$$

which, joined with (6.4), implies (1.15).

The proof of (1.16) is almost an immediate consequence. Here, however, a bit of care must be exercised. For instance, we will have summations of terms such as $C_{k,k_1,k_2,k_3} b_{j_1} b_{j_2} b_{j_3}(\sigma) 2^{-\sigma j_3}$ to control (see for instance Corollary 3.11), where $\{j_1, j_2, j_3\}$ is some permutation of $\{k_1, k_2, k_3\}$. Clearly such a term does not sum over $j_3 \ll C$. The way out, though, is straightforward. Recall that we need only sum over $Z_1(k), Z_2(k), Z_3(k)$, which are defined in (3.12). If we encounter a sum over $Z_1(k)$, bound the k_3 term with $b_{k_3}(\sigma) 2^{-\sigma k_3}$ and the remaining two terms with b_{k_1} and b_{k_2}. Over $Z_2(k)$, always bound the k_1 term by $b_{k_1}(\sigma) 2^{-\sigma k_1} \le b_{k_1}(\sigma) 2^{-\sigma k}$. Similarly, on $Z_3(k)$ we also bound the k_1 term by $b_{k_1}(\sigma) 2^{-\sigma k_1}$. Such a strategy suffices for controlling the perturbative terms. From here, though, it is only a small step to get $\sigma > 0$ bounds in §5.2, since we control the adapted forms B, $B_{\mathbf{e}}$ using the control on the perturbative terms.

We obtain

$$b_k(\sigma) \lesssim c_k(\sigma) + \varepsilon b_k(\sigma) + \sum_{j \le k - \varpi} (b_j c_j c_k(\sigma) + \varepsilon b_j^2 b_k(\sigma)),$$

which suffices to prove (1.16) in view of (6.5).

6.2. **Returning from the gauge to the map.** Equipped with Theorem 1.5, we show how to deduce Corollary 1.6.

To gain control over the derivatives $\partial_m \phi$ in $L_t^\infty L_x^2$, we utilize representation (2.2) and perform a Littlewood-Paley decomposition. We only indicate how to handle the term $v \cdot \mathrm{Re}(\psi_m)$, as the term $w \cdot \mathrm{Im}(\psi_m)$ may be handled

similarly. Starting with

$$
\begin{aligned}
P_k(v\mathrm{Re}(\psi_m)) = &\sum_{|k_2-k|\le 4} P_k(P_{\le k-5}v\cdot P_{k_2}\mathrm{Re}(\psi_m)) + \\
&\sum_{\substack{|k_1-k|\le 4\\ k_2\le k-4}} P_k(P_{k_1}v\cdot P_{k_2}\mathrm{Re}(\psi_m)) + \\
&\sum_{\substack{|k_1-k_2|\le 8\\ k_1,k_2\ge k-4}} P_k(P_{k_1}v\cdot P_{k_2}\mathrm{Re}(\psi_m)),
\end{aligned}
\tag{6.6}
$$

we proceed to bound each term in $L^\infty_t L^2_x$.

In view of the fact that $|v|\equiv 1$, the low-high frequency interaction is controlled by

$$
\begin{aligned}
\sum_{|k_2-k|\le 4}\|P_k(P_{\le k-5}v\cdot P_{k_2}\mathrm{Re}(\psi_m))\|_{L^\infty_t L^2_x} &\lesssim \|P_{\le k-5}v\|_{L^\infty_{t,x}}\|P_k\psi_m\|_{L^\infty_t L^2_x}\\
&\lesssim \|P_k\psi_m\|_{L^\infty_t L^2_x}\\
&\lesssim c_k.
\end{aligned}
\tag{6.7}
$$

To control the high-low frequency interaction, we use Hölder's inequality, Bernstein's inequality, (1.8) and Bernstein's inequality again, and finally the bound (2.16) along with summation rule (1.5):

$$
\begin{aligned}
\sum_{\substack{|k_1-k|\le 4\\ k_2\le k-4}}\|P_k(P_{k_1}v\cdot P_{k_2}\mathrm{Re}(\psi_m))\|_{L^\infty_t L^2_x} &\lesssim \sum_{\substack{|k_1-k|\le 4\\ k_2\le k-4}}\|P_{k_1}v\|_{L^\infty_t L^2_x}\|P_{k_2}\psi_m\|_{L^\infty_{t,x}}\\
&\lesssim \sum_{\substack{|k_1-k|\le 4\\ k_2\le k-4}}\|P_{k_1}v\|_{L^\infty_t L^2_x}\cdot 2^{k_2}\|P_{k_2}\psi_m\|_{L^\infty_t L^2_x}\\
&\lesssim \sum_{\substack{|k_1-k|\le 4\\ k_2\le k-4}}\|P_{k_1}\partial_x v\|_{L^\infty_t L^2_x}\cdot 2^{k_2-k}c_{k_2}\\
&\lesssim c_k.
\end{aligned}
\tag{6.8}
$$

To control the high-high frequency interaction, we use Bernstein's inequality, Cauchy-Schwarz, Bernstein again, (2.16), and finally (1.6):

$$\begin{aligned}
\sum_{\substack{|k_1-k_2|\le 8\\ k_1,k_2\ge k-4}} \|P_k(P_{k_1}v\cdot P_{k_2}\mathrm{Re}(\psi_m))\|_{L^\infty_t L^2_x} &\lesssim \sum_{\substack{|k_1-k_2|\le 8\\ k_1,k_2\ge k-4}} 2^k \|P_{k_1}v\cdot P_{k_2}\mathrm{Re}(\psi_m)\|_{L^\infty_t L^1_x}\\
&\lesssim \sum_{\substack{|k_1-k_2|\le 8\\ k_1,k_2\ge k-4}} 2^k \|P_{k_1}v\|_{L^\infty_t L^2_x}\|P_{k_2}\psi_m\|_{L^\infty_t L^2_x}\\
&\lesssim \sum_{\substack{|k_1-k_2|\le 8\\ k_1,k_2\ge k-4}} 2^{k-k_1} \|P_{k_1}\partial_x v\|_{L^\infty_t L^2_x}\|P_{k_2}\psi_m\|_{L^\infty_t L^2_x}\\
&\lesssim \sum_{k_2\ge k-4} 2^{k-k_2} c_{k_2}\\
&\lesssim c_k.
\end{aligned} \tag{6.9}$$

Combining (6.7), (6.8), and (6.9) and applying them in (6.6), we obtain

$$\|P_k(v\,\mathrm{Re}(\psi_m))\|_{L^\infty_t L^2_x} \lesssim c_k.$$

As the above calculation holds with w in place of v, we conclude (recalling (2.2)) that

$$\|P_k\partial_x\phi\|_{L^\infty_t L^2_x} \lesssim c_k. \tag{6.10}$$

Hence the result holds for $\sigma = 0$.

Now we turn to the case $\sigma \in (0, \sigma_1 - 1]$. By using Bernstein's inequality in (6.7) and (6.9), we may obtain

$$\sum_{|k_2-k|\le 4} \|P_k(P_{\le k-5}v\cdot P_{k_2}\mathrm{Re}(\psi_m))\|_{L^\infty_t L^2_x} \lesssim 2^{-\sigma k}c_k(\sigma) \tag{6.11}$$

$$\sum_{\substack{|k_1-k_2|\le 8\\ k_1,k_2\ge k-4}} \|P_k(P_{k_1}v\cdot P_{k_2}\mathrm{Re}(\psi_m))\|_{L^\infty_t L^2_x} \lesssim 2^{-\sigma k}c_k(\sigma), \tag{6.12}$$

as well as analogous estimates with w in place of v. Such a direct argument, however, does not yield the analogue of (6.8). The reason that there is an issue here even though there was not in the consideration N_k bounds on the Schrödinger nonlinearity has to do with the sort of right hand side bounds that appear. In controlling the Schrödinger nonlinearity, we had three terms $b_{\min}b_{\mathrm{mid}}b_{\max}$ at our disposal and we could always arrange inequalities so that $b_{\max}$ would be replaced by $2^{-\sigma k}b_k(\sigma)$, with k equal to the largest index of the three b_j's. Here we have only the one frequency envelope term c_k appearing (we could in fact make two appear in (6.8) if we first proved $\|P_k\partial_x v\|_{L^\infty L^2} \lesssim c_k$ by using (6.10) and then by essentially reversing the roles of v and ϕ in the argument). We circumvent this obstruction as follows.

Let $\mathcal{C} \in (0, \infty)$ be the best constant for which

$$\|P_k \partial_x \phi\|_{L_t^\infty L_x^2} \leq \mathcal{C} 2^{-\sigma k} c_k(\sigma) \tag{6.13}$$

holds for $\sigma \in [0, \sigma_1 - 1]$. Such a constant exists by smoothness and the fact that the $c_k(\sigma)$ are frequency envelopes. In view of definition (1.9) and estimate (2.29), we similarly have

$$\|P_k \partial_x v(0)\|_{L_t^\infty L_x^2} \lesssim \mathcal{C} 2^{-\sigma k} c_k(\sigma). \tag{6.14}$$

Using (6.14) in (6.8), we obtain

$$\sum_{\substack{|k_1 - k| \leq 4 \\ k_2 \leq k-4}} \|P_k(P_{k_1} v \cdot P_{k_2} \mathrm{Re}(\psi_m))\|_{L_t^\infty L_x^2} \lesssim \mathcal{C} 2^{-\sigma k} c_k c_k(\sigma). \tag{6.15}$$

From the representations (2.2) and (6.6), and from the estimates (6.11), (6.12), and (6.15), along with the analogous estimates for w, it follows that

$$\|P_k \partial_x \phi\|_{L_t^\infty L_x^2} \lesssim (1 + c_k \mathcal{C}) 2^{-\sigma k} c_k(\sigma).$$

In view of energy dispersion ($c_k \leq \varepsilon$) and the optimality of $\mathcal{C}$ in (6.13), we conclude

$$\mathcal{C} \lesssim 1 + \varepsilon \mathcal{C}$$

so that $\mathcal{C} \lesssim 1$. Therefore

$$\|P_k \partial_x^\sigma \partial_m \phi\|_{L_t^\infty L_x^2} \sim 2^{\sigma k} \|P_k \partial_m \phi\|_{L_t^\infty L_x^2} \lesssim c_k(\sigma).$$

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